Numerical Sequences

Series

Intuitive (of ideas) obtained by feelings rather than by considering facts:

Eventually # adv at the end of a period of time or a series of events or beyond.

Some point of time.

Primitive # belonging to an early stages in the development.

€—Neighbourhood

In any metric space (X, d) and $x_0 \in X$, open ball $B(x_0; \epsilon)$ is called an €—neighbourhood (or sometimes neighbourhood).

If $x_0 \in R$, then the open interval $(x_0 - \epsilon, x_0 + \epsilon)$ is called an €—neighbourhood of x_0. We denote neighbourhood by $N(x_0; \epsilon)$ or $N_\epsilon(x_0)$.

Deleted Neighbourhood

The set $(x_0 - \epsilon, x_0) \cup (x_0, x_0 + \epsilon)$ is called a deleted neighbourhood of x_0 because x_0

2

deleted from $N_\epsilon(x_0)$). It is denoted by $N_\epsilon'(x_0)$.

Sequence

A sequence in a set S is a function whose domain is the set N of natural numbers and range is a subset of the set S.

Real Sequence

A sequence of real numbers (or a sequence in R) is a function whose domain is the set N of natural numbers and range is a subset of the set R of real numbers.

Notations

(1) # If $x : N \longrightarrow R$ is a sequence, we usually denote the value of x at n by symbol x_n instead of the function notation $x(n)$. $x_1, x_2 \dots x_n, \dots$ are called terms of sequence.

(2). The sequence $x : N \longrightarrow R$ is denoted by $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$ or $< x_n >$ or $(x_n : n \in N)$ or (x_n).

(3) Notation $< x_n >$ or $(x_n : n \in N)$ or $\{(n, f(n)) : n \in N\}$ is used to that terms

he sequence ... an ordering
duced by the ... of natural n

The set o ... distinct terms
called the v ... and is denoted
$x_n : n \in N$ ... ues in the ran
t ordered.

Since in a ... nce $\{x_n\}, n \in$
s an infinite ... therefore the

terms of a sequ ... is always infin
range of sequen ... ay be a finite
if $x_n = (-1)$ ... $\{x_n\} = \{-1, 1$
The range f ... nce $= \{-1, 1, 3$

5) # sequences ... often defined
a formula for ... nth term x_n

7) # Sometimes ... convenient to
terms of a seq ... in order st
the rule of ... tion is perm
nd nth term

8) # The m ... reated distin
for $m \neq n$ a ... curring at di
ie the terms ... distinct term
are treated ... value
they have

4

Constant Sequence

A sequence $\{x_n\}$ defined by $x_n = c \in R$ $\forall n \in N$ is called a constant sequence.

OR

A sequence whose range is a singleton is called a constant sequence.

Mathematical Formula Notation

For Sequence

Many sequences may be defined by some mathematical rule by two ways.

(a) By an explicit formula. (b) by a recursion.
or inductive or iterative formula.

(a)

Explicit Formula

A sequence may be defined by giving an explicit formula for the nth term. e.g

(1) $a_n = \frac{1}{n}$ (2) $a_n = \frac{n}{n+1}$

(3) $a_n = (-1)\frac{n}{n+1}$ $a_n = 3$

(b)

Recursive Formula

Sometimes sequences are defined by specifying one or more initial terms and (clearly giving) a formula that relates each subsequent by giving a term to the previous terms. Such (next coming)

5

Sequences are said to be defined recursively or inductively or iteratively and the defining formula is called a recursion formula or recursion formula.

OR

A sequence defined by a formula for the nth term in terms of one or several previous terms with some initial terms specified clearly.

Examples

$\boxed{1}$

$a_{n+2} = a_{n+1} + a_n \qquad a_1 = 1 \quad a_2 = 1$

$1, 1, 2, 3, 5, 8 \cdots$

2) #

$a_n = \dfrac{a_{n-1}}{2} \qquad n > 1 \qquad a_0 = 1$

$a_1 = 1$

3) #

$a_n = n \cdot a_{n-1}$

Fibonacci Sequence.

4) #

$a_{n+1} = a_n + a_{n-1} \qquad a_1 = 1, \quad a_2 = 1$

$1, 1, 2, 3, 5, 8 \cdots$

Thus each term after the 1st two terms is the sum of its two immediate previous terms.

5) The sequence $\{x_n\}$ of even numbers can be defined by $x_1 = 2 \qquad x_{n+1} = x_n + 2$.

OR by $y_1 = 2 \qquad y_{n+1} = y_1 + y_n$.

6

Note not every sequence has a formula. or even any formula at all. e.g $\{a_n\}$, where a_n is nth digit in its decimal representation of π.

Bounded Above Sequence.

A sequence $\{x_n\}$ is said to be bounded above if $\exists$ a real no K such that

$$a_n \leq K \qquad \forall n \in N$$

i.e if the range of the sequence is bounded above.

Bounded Below Sequence.

A sequence $\{x_n\}$ is said to be bounded below if $\exists$ a real no k such that

$$a_n \geqslant k \qquad \forall n \in N.$$

i.e if the range of the sequence is bounded below.

Bounded Sequence

A sequence is said to be bounded if it is bounded above as well as below:

Thus a sequence $\{x_n\}$ is bounded if $\exists$ two numbers k & K ($k \leq K$) such that

22

i.e if $k \leq x_n \leq k$ $\forall n \in N$.

i.e if the range set $\{x_n : n \in N\}$ is bounded.

A sequence that is not bounded is said to be unbounded sequence.

Unbounded Above

A sequence $\{x_n\}$ is said to be unbounded above if it is not bounded above i.e if for every real no K $\exists$ m∈N s.t that

$$a_m > k.$$

Unbounded Below

A sequence $\{x_n\}$ is said to be unbounded below if it is not bounded below i.e if for every real no K $\exists$ m∈N s.t

$$a_m < k$$

Examples

(1) # The sequence $\{a_n\}$ defined by $a_n = \frac{1}{n}$ is bounded. because. $0 < a_n \leq 1$.

(2) # The sequence $\{a_n\}$ defined by $a_n = n$ $\forall n \in N$. is bounded below by 1 because $a_n \geq 1$ $\forall n \in N$. It is not bounded above because $\exists$ no real no K such that $a_n \leq k$ $\forall n \in N$

(3) # The sequence $\left\{\frac{8}{(-1)^n}\right\}$ is bounded because.

$$-1 \leq a_n \leq 1 \qquad \forall \, n \in N$$

(4) # The sequence $\{-n\}$ is bounded above

because $a_n \leq -1 \quad \forall \, n \in N$ & is not

bounded below

(5) # Every constant sequence is bounded.

(6) # The sequence $\{a_n\}$ defined by $a_n = (-1)^n \cdot n$

is neither bounded above nor bounded below.

Theorem

A sequence $\{a_n\}$ is bounded

iff $\exists$ a +ve real no M such that

$$|a_n| \leq M \qquad \forall \, n \in N.$$

Proof

Necessary Condition.

Let $\{a_n\}$ be bounded. The $\exists$ two

real numbers h & k such that

$$h \leq a_n \leq k \qquad \forall \, n \in N.$$

Let $M = \text{Max}\{|h|, |k|\}$. Then.

$$|h| \leq M \qquad \& \qquad |k| \leq M.$$

$$\Rightarrow -M \leq h \leq M \qquad \& \qquad -M \leq k \leq M.$$

$$\Rightarrow -M \leq h \leq a_n \leq k \leq M \qquad \forall \, n \in N$$

$$\Rightarrow -M \leq a_n \leq M \qquad \forall \, n \in N$$

$$\Rightarrow |a_n| \leq M \qquad \forall \, n \in N.$$

Sufficient Condition#

Let M be the real no such that

$$|a_n| \leq M \quad \forall n \in N.$$

Then $-M \leq a_n \leq M \quad \forall n \in N.$

$\Rightarrow \{a_n\}$ is bounded.

Note The above Theorem is used as definition of bounded sequence.

Theorem# If $\{a_n\}$ & $\{b_n\}$ are bounded sequences and c is a real no, then

(a) $\{a_n + b_n\}$ is bounded.

(b) $\{a_n b_n\}$ is bounded.

(c) $\{c a_n\}$ is bounded.

Proof#

$\because \{a_n\}$ & $\{b_n\}$ are bounded.

$\therefore \exists$ +ve numbers M_1 & M_2

such that $|a_n| \leq M_1$ & $|b_n| \leq M_2 \quad \forall n \in N.$

$|a_n| + |b_n| \leq M_1 + M_2 = M \quad \forall n \in N$

$\Rightarrow |a_n + b_n| \leq |a_n| + |b_n| \geq M \quad \forall n \in N.$

$\Rightarrow \{a_n + b_n\}$ is bounded.

$|a_n b_n| = |a_n| |b_n| \leq M_1 M_2 = M \quad \forall n \in N$

$\Rightarrow \{a_n b_n\}$ is bounded.

(c) # $|ca_n|$

$= |c||a_n| > k|M_1 = M_4$ $\forall n$

$\Rightarrow \{a_n\}$ is bounded.

Eventual Property of a Sequence.

If the terms of a sequence do not have a certain property from start but have that property from some point (from some term) on, then the sequence has that property eventually.

If a sequence $\{a_n\}$ fulfil some property P eventually, then mathematically we say that

$\exists$ an integer $n_1 \in \mathbb{N}$ such that

if $n \geqslant n_1$, $\{a_n\}$ satisfies property P.

Limit of a Sequence & Convergence

A sequence $\{a_n\}$ in $\mathbb{R}$ is said to converge to $l \in \mathbb{R}$ or l is a limit of $\{a_n\}$ if for every $\epsilon > 0$ $\exists$ a natural no $n_1(\epsilon)$ such that

$$|a_n - l| < \epsilon \qquad \forall n \geqslant n_1$$

If so we write $\lim\limits_{n \to \infty} a_n = l$ or $\lim a_n = l$.

or $a_n \to l$.

If a sequence has a limit, then sequence is convergent, if it has no limit, the sequence is dgt.

2.1

Note (1) A real no l is a limit of sequence $\{a_n\}$ if given $\epsilon > 0$, all but a finite no of terms of $\{a_n\}$ lie within ϵ of l

(2)
$$\forall n \geq n_1$$
$$|a_n - l| < \epsilon$$
$$\Rightarrow l - \epsilon < a_n < l + \epsilon \qquad \forall n \geq n_1$$
$$\Rightarrow l - \epsilon < \qquad \qquad l + \epsilon) \qquad \forall n \geq n_1$$
$$\Rightarrow a_n \in (l - \epsilon, \ l + \epsilon) \qquad \forall n \geq n_1$$

$\Rightarrow$ given any $\epsilon > 0$, all the terms of the sequence, except the 1st $n_1 - 1$ terms lie in the interval $(l - \epsilon, \ l + \epsilon)$. The 1st $n_1 - 1$ terms may be scattered anywhere. The number $n_1 - 1$ of terms left out of the interval $(l - \epsilon, \ l + \epsilon)$ depends upon the size of ϵ. The smaller the size of ϵ, the larger will be the no of terms left out of $(l - \epsilon, \ l + \epsilon)$.

(2) We say that a sequence $\{a_n\}$ ultimately has a certain property if $\exists$ a no n_1 such that sequence $\{a_n\}$ satisfies that property for $n \geq n_1$. A sequence $\{a_n\}$ converges to l if the terms of $\{a_n\}$ are ultimately in every $\epsilon-$ neighbourhood of l.

(3)

With the Language of Nbhd.

A sequence $\{a_n\}$ converges to number l if for each ϵ-nbhd $N_\epsilon(l)$ of l all but a finite no of terms of $\{a_n\}$ belong to $N_\epsilon(l)$

Examples

(a) $\lim_{n \to \infty} (1/n) = 0$

Let $a_n = \frac{1}{n}$ and $\epsilon > 0$ be given.

$|a_n - l| = |\frac{1}{n} - 0| = \frac{1}{n}$.

and $|a_n - l| < \epsilon$ if $\frac{1}{n} < \epsilon$.

if $n > \frac{1}{\epsilon}$.

Thus if we take natural no n_1 greater than real no $\frac{1}{\epsilon}$, then we have.

$|a_n - l| < \epsilon \qquad \forall n \geqslant n_1.$

$\Rightarrow a_n \longrightarrow 0$

Explaination:

For each $\epsilon > 0$, we can always trace n_1 by relation $n > \frac{1}{\epsilon}$ let us see how.

for $\varepsilon = .1$

$\dfrac{1}{\varepsilon} = \dfrac{1}{.1} = 10$

our n_1 will be greater than 10 because for $a_{10} = \dfrac{1}{10}$ we have

$|a_{10} - 0| = |\dfrac{1}{10} - 0| = .1$

$a_{11} = \dfrac{1}{11}$

$|a_{11} - 0| = \dfrac{1}{11} = .09 < = .1$

Thus for $\varepsilon = .1$, $n_1 = 11$ &

$|a_n - 0| < \varepsilon \quad \forall\, n \geq n_1$

for $\varepsilon = .01$

$\dfrac{1}{\varepsilon} = \dfrac{1}{.01} = 100$

n_1 will be 101 or greater

and

$|a_n - 0| < \varepsilon = .01 \quad \forall\, n \geq 101 = n_1$

We note that for smaller ε, the greater n_1

For $\varepsilon = .5$

$\dfrac{1}{\varepsilon} = \dfrac{1}{.5} = 2$

n_1 will be 3 or greater
and.

$|a_n - 0| < \varepsilon = .5 \quad \forall\, n \geq 3 = n_1$

We note that for greater ε, the smaller n_1

(b) $\operatorname{Lim} \dfrac{1}{n^2+1} = 0$

let $a_n = \dfrac{1}{n^2+1}$ $\ell = 0$ let $\varepsilon > 0$ be

given

$|a_n - \ell| = |\dfrac{1}{n^2+1} - 0| = \dfrac{1}{n^2+1} \leq \dfrac{1}{n^2} \leq \dfrac{1}{n}$

Thus if $\dfrac{1}{n} < \varepsilon$ or $n > \dfrac{1}{\varepsilon}$

we have for n, greater than $\frac{14}{\epsilon}$

$|a_n - \ell| < \epsilon$ ∀ n ≥ n₁.

⇒ $a_n \longrightarrow 0$

• (c) $\lim_{n\to\infty}\left(\frac{3n+2}{n+1}\right) = 3$

Let $a_n = \frac{3n+2}{n+1}$ $\ell = 3$ & ε>0 be given.

$|a_n - \ell| = \left|\frac{3n+2}{n+1} - 3\right| = \left|\frac{3n+2-3n-3}{n+1}\right|$

$= \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \frac{1}{n}$

$< \epsilon$ if $\frac{1}{n} < \epsilon$.

if $n > \frac{1}{\epsilon}$

Thus $|a_n - \ell| < \epsilon$ if $n > \frac{1}{\epsilon}$

So we can take n_1 greater than $\frac{1}{\epsilon}$
for each ε>0 such that

$|a_n - \ell| < \epsilon$ ∀ n ≥ n₁.

⇒ $a_n \longrightarrow \ell = 3$

(d) $\lim_{n\to\infty}\left(\sqrt{n+1} - \sqrt{n}\right) = 0$

$\sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$

$= \frac{n+1 - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$|a_n - \ell| = \left|\frac{1}{\sqrt{n+1} + \sqrt{n}} - 0\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}}$

$= \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}}$

33

if $\frac{1}{\sqrt{n}} < \epsilon$, then $n > \frac{1}{\sqrt{\epsilon}}$

Thus for each $\epsilon > 0$, if we take n greater than $\frac{1}{\sqrt{\epsilon}}$, we have.

$|a_n - \ell| < \epsilon$ $\forall n \geq n_1$

$\Rightarrow a_n \longrightarrow \ell = 0$

(e) $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$.

$a_n = \frac{n^2-1}{2n^2+3}$ $\ell = \frac{1}{2}$ let $\epsilon > 0$

$|a_n - \ell| = \left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right|$

$= \left|\frac{2n^2-2-2n^2-3}{2(2n^2+3)}\right|$

$= \frac{5}{4n^2+6} < \frac{5}{4n^2} < \frac{5}{n^2}$

$|a_n - \ell| < \epsilon$

if $\frac{5}{n^2} < \epsilon$

if $\frac{5}{\epsilon} < n^2$

if $n > \sqrt{\frac{\epsilon}{5}}$

$\Rightarrow$ If we take n, greater than $\sqrt{\frac{\epsilon}{3}}$, then

$|a_n - \ell| < \epsilon$ $\forall n \geq n_1$

$\Rightarrow a_n \longrightarrow \ell = \frac{1}{2}$.

(f) If $0 < b < 1$, then $\lim b^n = 0$

For any $a > 0$, we can write.

$b = \frac{1}{1+a} < 1$.

$\Rightarrow a = \frac{1}{b} - 1 > 0$ $\therefore b < 1$

$$a+1 = \frac{1}{b}$$

Now By Bernoulli's Inequality

$$(1+a)^n \geq 1+na.$$

Hence.

$$0 < b^n = \frac{1}{(a+1)^n} \leq \frac{1}{1+na} < \frac{1}{na}.$$

$$|a_n - 0| = |b^n - 0| = b^n < \frac{1}{na}.$$

$$|a_n - 0| < \epsilon \quad \text{if} \quad \frac{1}{na} < \epsilon$$

$$\text{if} \quad \frac{1}{n} < a\epsilon$$

$$\text{if} \quad n > \frac{1}{a\epsilon}$$

$\Rightarrow$ we can take $n_1 > \frac{1}{a\epsilon}$ for each $\epsilon > 0$

such that

$$|a_n - 0| < \epsilon \quad \forall n \geq n_1$$

OR

$$|a_n - 0| = b^n$$

$$|a_n - 0| < \epsilon \quad \text{if} \quad b^n < \epsilon$$

$$\text{if} \quad n\ln b < \ln \epsilon$$

$$\text{if} \quad n > \frac{\ln \epsilon}{\ln b} \qquad \therefore \ln b < 0$$

Thus if we choose no $n_1 > \frac{\ln \epsilon}{\ln b}$, we have

$$|a_n - 0| < \epsilon \qquad \forall n > n_1$$

e.g if $b = .9$ & if $\epsilon = .01$. we have $n_1 > \frac{\ln .01}{\ln 8}$

≈ 20.6377. Thus $n_1 = 21$ would be appropriate for $\epsilon = .01$

17
Uniqueness of Limit

Theorem# A sequence in R can have at most one limit.

OR

Every convergent sequence in R has a unique limit

OR

The limit of the sequence, if it exists, is unique.

Proof# Let $\{a_n\}$ be an arbitrary convergent sequence and A, B be two limits of the sequence.

Suppose that- $A \neq B$

Now $\dfrac{|A-B|}{2} > 0$

∵ $\{a_n\}$ Converges to A

∴ For $\epsilon = \dfrac{|A-B|}{2}$ ∃ a natural no n_1 such that

$$|a_n - A| < \dfrac{|A-B|}{2} \qquad n_1 < n \to ①$$

∵ $\{a_n\}$ Converges to B

∴ for $\epsilon = \dfrac{|A-B|}{2}$ ∃ a natural no n_2 such that

$$|a_n - B| < \dfrac{|A-B|}{2} \qquad n_2 < n \to ②$$

Let $n_3 = Max\{n_1, n_2\}$

Then

$\Rightarrow |a_n - A| < \dfrac{|A-B|}{2}$ 18 $\forall n \geq n_3 \longrightarrow$ ③

and $|a_n - B| < \dfrac{|A-B|}{2}$ $\forall n \geq n_3 \longrightarrow$ ④

$$|A-B| = |A - a_n + a_n - B|$$
$$\leq |A - a_n| + |a_n - B|$$
$$< \frac{|A-B|}{2} + \frac{|A-B|}{2} \qquad \forall n \geq n_3$$

$\Rightarrow |A-B| < |A-B|$

which is absurd. Hence $A = B$

$\Rightarrow$ limit is unique.

OR

$\therefore \displaystyle\lim_{n \to \infty} a_n = A \qquad \lim_{n \to \infty} a_n = B$

$\therefore$ For any $\epsilon > 0$ $\exists$ n_1 & n_2 such that

$|a_n - A| < \epsilon/2 \qquad \forall n \geq n_1$

$|a_n - B| < \epsilon/2 \qquad \forall n \geq n_2$

Let $n_3 = \max(n_1, n_2)$

Then $|a_n - A| < \epsilon/2. \qquad \forall n \geq n_3$

$|a_n - B| < \epsilon/2 \qquad \forall n \geq 3$

$|A - B| = |A - a_n + a_n - B|.$

$|A-B| \leq |A-a_n| + |a_n - B|$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall n \geq n_3$

$\Rightarrow |A-B| < \epsilon \qquad \forall n \geq n_3$

$\therefore \epsilon$ is an arbitrary true quantity

$\therefore |A-B|$ is less than every true quantity however small and so must be zero

Thus $\quad |A-B| = 0$

$\Rightarrow |A-B| = 0$

$\Rightarrow A - B = 0 \Rightarrow A = B$

$\Rightarrow$ Limit of sequence is unique.

$\underline{\underline{Theorem}}$#

Let $\{x_n\}$ be a sequence of real numbers and let $x \in R$. The Following statements are equivalent

(a)# $\{x_n\}$ converges to x.

(b)# For every $\epsilon > 0$, $\exists$ a natural no K. such that

$\qquad |x_n - x| < \epsilon \qquad \forall n \geq K$.

(c)# For every $\epsilon > 0$, $\exists$ a natural no. K. s. that $\quad x - \epsilon < x_n < x + \epsilon \qquad \forall n \geq K$.

(d)# For every neighbourhood $V_\epsilon (x)$ of x $\exists$ a natural no K such that $\forall n \geq K$, $x_n \in V_\epsilon (x)$

(a) $a \Rightarrow b \Rightarrow c \Rightarrow d$

Let $\{x_n\}$ be convergent to x.

Then by definition for every $\varepsilon > 0$ $\exists$ a natural no k such that

$$|x_n - x| < \varepsilon \qquad \forall n > k.$$

$\Rightarrow x - \varepsilon < x_n < x + \varepsilon \qquad \forall n > k.$

$\Rightarrow x_n \in (x-\varepsilon, \ x+\varepsilon) \qquad \forall n > k.$

$\Rightarrow$ for every ε-nbhd $V_\varepsilon(x)$ $\exists$ a natural no k such that

$$x_n \in V_\varepsilon(x) \qquad \forall n > k.$$

Divergent Sequence#

(a) A sequence $\{a_n\}$ is said to diverge to $+\infty$ if given any +ve real no k, however large, $\exists$ a natural no n_1 such that

$$a_n > k \qquad \forall n > n_1$$

and we write

$$\lim_{n \to \infty} a_n = \infty \quad \text{or} \quad a_n \to \infty \text{ as } n \to \infty$$

(b) A sequence $\{a_n\}$ is said to diverge to $-\infty$ if given any +ve real no k,

33

however large, $\exists$ a natural no n_1.
such that $a_n < -k$ $\forall n \geq n_1$

We write $\lim_{n\to\infty} a_n = -\infty$ or $a_n \to -\infty$

Equivalently a sequence $\{a_n\}$ diverges to $-\infty$ if given any $-ve$ real no k $\exists$ a natural no n_1 s.t
$a_n < k$ $\forall n \geq n_1$

(c) A sequence $\{a_n\}$ is said to be divergent sequence if it diverges to $+\infty$ or $-\infty$

Examples

(i) The sequences $\{n\}$ & $\{n^2\}$ diverge to ∞

(ii) The sequence $\{-n\}$ & $\{-n^2\}$ diverge to $-\infty$

Oscillatory Sequence

If a sequence $\{a_n\}$ neither converges to a finite number nor diverges to $+\infty$ or to $-\infty$, it is called an oscillatory sequence.

22

Oscillatory sequences are of Two types

(a) A bounded sequence which does not Converge is said to oscillate finitely.

e.g $\{(-1)^n\}$

(b) Here $a_n = (-1)^n$ $\qquad a_{2n} = (-1)^{2n} = 1$

$\qquad a_{2n+1} = (-1)^{2n+1} = -1$

$\displaystyle \lim_{n \to \infty} a_{2n} = 1 \qquad \lim_{n \to \infty} a_{2n+1} = -1$

$\Rightarrow \displaystyle \lim a_n$ does not exist $\Rightarrow$ sequence does not Converge. It is bounded because

$|a_n| = 1$

Hence sequence oscillate finitely

Hence sequence oscillate sequence which does

(b) An unbounded sequence is said to oscillate not diverge is said to oscillate

infinitely. e.g $\{(-1)^n n\}$

$a_n = (-1)^n n$

$\displaystyle \lim_{n \to \infty} a_{2n} = \lim (2n) = \infty$

$\displaystyle \lim a_{2n+1} = \lim (-1)^{2n+1}(2n+1) = -\infty$

Thus sequence does not diverge

Hence this sequence oscillates infinitely

Null Sequence

A sequence which converges to zero is said to be a null sequence.

e.g $\left\{\frac{1}{n}\right\}, \left\{\frac{1}{n^2}\right\}, \left\{\frac{1}{2n}\right\} \{ \} \left\{\frac{(-1)^{n-1}}{n}\right\}$

are null sequences

Note: A sequence $\{a_n\}$ is called infinitely small if $\lim a_n = 0$ & infinitely large if $\lim a_n = \infty$

Theorem

Let $\{x_n\}$ be a sequence of real numbers and $x \in R$.

If $\{a_n\}$ is a sequence of two real nos. with $\lim a_n = 0$ and if for some $c > 0$ and some natural no n_1, we have

$$|x_n - x| \leq c a_n \qquad \forall n \geq n_1$$

Then it follows that $\lim_{n \to \infty} (x_n) = x$

Proof: $\because \lim_{n \to \infty} a_n = 0$

$\therefore$ For given $\epsilon > 0$ $\exists$ exists a natural no $k(\epsilon/c)$ such that

$$|a_n - 0| < \epsilon/c \qquad \forall n \geq k.$$

$$\Rightarrow a_n < \epsilon/c \qquad \forall n \geq k.$$

24

Let $n_2 = Max(k, n_1)$

Then $|x_n - x| \le c a_n$ $\qquad \forall n \ge n_2$ →①

and $a_n < \epsilon/c$ $\qquad \forall n \le n_2$ →②

By ① and ②

$|x_n - x| \le c a_n < c \left(\epsilon/c\right) = \epsilon$ $\qquad \forall n \ge n_2$.

Since ϵ is arbitrary, we have

$$\lim_{n \to \infty} x_n = x$$

Examples

1)# If $a > 0$, then $\lim\left(\dfrac{1}{1+na}\right) = 0$

Sol# $\because$ $a > 0$

$\therefore$ $0 < na < 1 + na$.

$\Rightarrow 0 < \dfrac{1}{1+na} < \dfrac{1}{na} = \dfrac{1}{a}\left(\dfrac{1}{n}\right)$.

Thus $\left|\dfrac{1}{1+na} - 0\right| \le \left(\dfrac{1}{a}\right)\dfrac{1}{n}$ $\qquad \forall n \in N$.

Since $\lim \dfrac{1}{n} = 0$ and $\dfrac{1}{a} > 0$ & $n = 1$

from above theorem we have

$$\lim_{n \to \infty}\left(\dfrac{1}{1+na}\right) = 0$$

2)# If $0 < b < 1$, then $\lim_{n \to \infty} b^n = 0$

Sol $\because$ $0 < b < 1$

25

$\therefore \ b = \dfrac{1}{1+a} \Rightarrow a = \dfrac{1}{b} - 1$

So $a > 0$ $(\because \ b < 1)$

By Bernoulli's Inequality, we have

$(1+a)^n \geqslant 1+na$

Hence

$0 < b^n = \dfrac{1}{(1+a)^n} \leqslant \dfrac{1}{1+na} < \dfrac{1}{na}$

$\Rightarrow b^n < \left(\dfrac{1}{a}\right) \dfrac{1}{n}$ $\forall n \geqslant 1$.

$\therefore \ \lim\limits_{n \to \infty} \dfrac{1}{n} = 0$ & $\dfrac{1}{a} = c > 0$

$\therefore$ By above theorem.

$\lim\limits_{n \to \infty} b^n = 0$

3) If $c > 0$, then $\lim\limits_{n \to \infty} c^{1/n} = 1$

__Case I__ If $c = 1$, then sequence $\{c^{1/n}\}$
is constant sequence $\{1,1,1,\ldots\}$ which.
Converges to 1.

__Case II__ If $c > 1$, then $c^{1/n} = 1 + d_n$, some $d_n > 0$

By Bernoulli's Inequality

$(1+d_n)^n \geqslant 1+nd_n$ $\forall n \in N$

$(1+1) \geqslant 1+nd_n$

$c = $

$c-1 \geqslant nd_n$

$\Rightarrow \ d_n \leqslant \dfrac{c-1}{n}$ $\forall n \in N$.

$\Rightarrow \ d_n \leqslant \dfrac{c-1}{n}$ $\forall n \in N$

now $|c^{1/n} - 1| = d_n \leqslant (c-1) \dfrac{1}{n}$ $\forall n$ $\{c-1 > 0 \ \because \ c > 1\}$

By above theorem.

$$\lim_{n\to\infty} c^{1/n} = 1$$

<u>Case III</u> If $0 < c < 1$, then $c^{1/n} = \frac{1}{1+h_n}$ for some $h_n > 0$. By Bernoulli's Inequality we have

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} < \frac{1}{nh_n}$$

$\Rightarrow \quad 0 < h_n < \frac{1}{nc} \qquad \forall n \in N.$

$0 < 1 - c^{1/n} = \frac{h_n}{1+h_n} < h_n < \frac{1}{nc} \qquad \forall n \in N.$

$\Rightarrow \quad |c^{1/n} - 1| < \left(\frac{1}{c}\right)\frac{1}{n} \qquad \forall n \in N.$

By above theorem.

$$\lim_{n\to\infty} c^{1/n} = 1$$

4) $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore \quad n^{1/n} > 1 \qquad \forall n > 1$

<u>Sol</u>

Let $n^{1/n} = 1 + k_n$ for some $k_n > 0$

$\Rightarrow \quad n = (1+k_n)^n \qquad \forall n > 1$

$= 1 + nk_n + \frac{1}{2}n(n-1)k_n^2 + \dots \geq \frac{1}{2}n(n-1)k_n^2$

$\Rightarrow \quad n \geq \frac{n(n-1)}{2}k_n^2 + 1$

$$-1 \geq \frac{n(n-1)}{2} x_n^2$$

$$x_n^2 \leq \frac{2}{n} \qquad \forall n>1$$

$$x_n \leq \sqrt{\frac{2}{n}} \qquad \forall n>1$$

$$0 < n^{1/n} - 1 = x_n \leq \sqrt{\frac{2}{n}} \qquad \forall n>1$$

$$|n^{1/n} - 1| \leq \sqrt{\frac{2}{n}} \qquad \forall n>1$$

By above theorem

$$\lim_{n \to \infty} n^{1/n} = 1$$

Theorem # Every Convergent sequence is bounded.

Proof # Let $\{a_n\}$ be an arbitrary convergent sequence of real number and.

$$\lim_{n \to \infty} a_n = A$$

For $\epsilon = 1$ $\exists$ a natural no m

that $\quad |a_n - A| < 1 \quad \forall n \geq m \quad \longrightarrow ①$

$$|a_n| = |a_n - A + A|$$

$$\leq |a_n - A| + |A|$$

$$< 1 + |A| \qquad \forall n \geq m$$

Also

$a_n \leq K \quad \forall n \geq m-1$

and $a_n < A+1 \leq K \quad \forall n \geq m$

$\Rightarrow a_n \leq K \quad \forall n$ ... (5)

By ④ & ⑤

$k \leq a_n \leq K \quad \forall n$

$\Rightarrow \{a_n\}$ is bounded.

Discussion #(1) The intuition for this theorem is
quite simple. First by definition of limit all
the terms with large index must be close to A).
Since the no of terms with small index is finite
, and every finite set is bounded, therefore we
are able to construct a bound for all terms of
the sequence.

(2)# The Converse of its above theorem.
is not true i.e a bounded sequence is not necessarily
Convergent e.g the sequence $\{(-1)^n\}$ is bounded.

(3) Convergence $\longrightarrow$ Boundedness
Contra-positive if it is

A sequence that is not bounded can never
Converge.

This is a useful tool for showing certain sequences
do not Converge. The sequence $\{n\}$ diverges
because the set of the integers is not bounded.

Theorem# If a sequence is unbounded, then it must diverge or it can not converge.

Proof# Suppose that the sequence $\{a_n\}$ is unbounded and let on the contrary it converges and its limit is A.

Then for $\varepsilon = 1$, $\exists$ a natural no m.

such that

$$|a_n - A| < 1 \qquad \forall n \geq m.$$

$$|a_n| = |a_n - A + A| \leq |a_n - A| + |A| \leq 1 + |A|$$

$$|a_n| \qquad \forall n \geq m$$

if $n > m$, then

$$|a_n| \leq \text{Max}\{|a_1|, |a_2|, \ldots , |a_{m-1}|\}$$

Let $M = \text{Max}\{|a_1|, |a_2|, \ldots , |a_{m-1}|, 1 + |A|\}$

Then $|a_n| \leq M \qquad \forall n \in N$.

$\Rightarrow \{a_n\}$ is bounded. which is a contradiction.

Hence $\{a_n\}$ is not convergent.

Theorem# If $\{a_n\}$ is a convergent sequence of real numbers such that $a_n \geq 0$
and $\lim\limits_{n \to \infty} a_n = A$, then $A \geq 0$.

Proof# Let on contrary $A < 0$

31

Then $-A > 0$

$\therefore \lim\limits_{n\to\infty} a_n = A$

$\therefore$ For $\epsilon = -A > 0$ $\exists$ a natural no n_1 such that

$|a_n - A| < -A = \epsilon \quad \forall n \geq n_1$

$\Rightarrow A - \epsilon < a_n < A + \epsilon \quad \forall n \geq n_1$

In particular, we have

$a_{n_1} < A + \epsilon = A + (-A) = 0$

$\Rightarrow a_{n_1} < 0$

But this contradicts the the hypothesis that $a_n \geq 0 \ \forall n$. Thus $A \geq 0$

OR

For $\epsilon = -\dfrac{A}{2} > 0$ $\exists$ a natural no n_1 such that $|a_n - A| < -\dfrac{A}{2} = \epsilon \quad \forall n \geq n_1$

$\Rightarrow a_n - A > -\dfrac{A}{2}$

$a_n > \dfrac{A}{2} \quad \forall n \geq n_1$

$\Leftarrow a_{n_1} < \dfrac{A}{2} < 0$

$\Rightarrow a_{n_1} < 0$

But by hypothesis $a_n \geq 0 \ \forall n$

a contradiction. Thus $A \geq 0$

Hence a contradiction. Thus $A \geq 0$

Note(1) This theorem states that a the sequence of non-negative terms if converges, then converges

10.

to a non-negative limit. $\underline{32}$

(2) If the sequence ultimately becomes non-negative, then if exists, it will have non-negative limit.

Theorem # of a sequence $\{a_n\}$ Converges
to A, then the sequence $\{|a_n|\}$ Converges to $|A|$

Proof #: $\because \{a_n\}$ Converges to A

$\therefore$ For a given $\epsilon > 0$ $\exists$ a natural

no n_1 such that

$$|a_n - A| < \epsilon \qquad \forall n \geq n_1$$

Now $||a_n| - |A|| \leq |a_n - A| < \epsilon \qquad \forall n \geq n_1$

$\Rightarrow \{|a_n|\}$ Converges to $|A|$

$\Rightarrow$ Converse of above is not true $|a_n| = 1$ gt but $\{(-1)^n\}$ is dgt

Theorem Let $\{x_n\}$ be a sequence of real nos,
that converges to x and suppose that $x_n \geq 0$.
then the sequence $\{\sqrt{x_n}\}$ of +ve square roots Converges

and $\lim\limits_{n \to \infty} (\sqrt{x_n}) = \sqrt{x}$

Proof # $\because x_n \geq 0$ $\therefore \lim\limits_{n \to \infty} x_n \geq 0$

So the theorem makes the sense.

Case (i) if $x = 0$, then $x_n \to 0$ and for
given $\epsilon > 0$ $\exists$ a natural no n_1 such that

$$|x_n - 0| < \epsilon^2 \qquad \forall n \geq n_1$$

$\Rightarrow x_n \geq 0$

$\Rightarrow \sqrt{x_n} \geq 0$

$\therefore \in$ is arbitr...

Case ii: if $x >$

$\sqrt{x_n} - \sqrt{x} =$

$\sqrt{x_n} + \sqrt{x} \geq$

$\therefore \dfrac{1}{\sqrt{x_n} + \sqrt{x}} \leq$

and

$\sqrt{x_n} - \sqrt{x} = \dfrac{x_n -}{\sqrt{x_n}}$

$\Rightarrow |\sqrt{x_n} - \sqrt{x}| \leq \dfrac{|}{}$

$\therefore x_n \longrightarrow x$

$\therefore$ for $\in \sqrt{x} > 0$

such that $|x_n - x|$

using in ①

$|\sqrt{x_n} - \sqrt{x}| \leq$

34

Theorem # Let $\{a_n\}$ be a sequence.

$\lim\limits_{n\to\infty} a_n = 0$ iff $\lim\limits_{n\to\infty}|a_n| = 0$ i.e $\{a_n\}$ is a null sequence iff $\{|a_n|\}$ is a null sequence.

Proof # Suppose that $\{a_n\}$ is a null sequence, then $\lim\limits_{n\to\infty} a_n = 0$

∴ Given $\varepsilon > 0$ ∃ a +ve integer n_1 such that

$$|a_n - 0| < \varepsilon \qquad \forall\, n \geq n_1$$

$$\Rightarrow |a_n| < \varepsilon \qquad \forall\, n \geq n_1$$

$$\Rightarrow \big||a_n| - 0\big| = \big||a_n|\big| = |a_n| < \varepsilon \qquad \forall\, n \geq n_1$$

$$\Rightarrow \{|a_n|\} \text{ is a null sequence.}$$

Now Suppose that $\{|a_n|\}$ is a null sequence, then $\lim\limits_{n\to\infty}|a_n| = 0$

∴ Given $\varepsilon > 0$ ∃ a natural no m such that

$$\big||a_n| - 0\big| < \varepsilon \qquad \forall\, n < A$$

$$\Rightarrow \big||a_n|\big| < \varepsilon \qquad \forall\, n < A$$

$$\Rightarrow |a_n| < \varepsilon \qquad \forall\, n < A$$

$$\Rightarrow |a_n - 0| < \varepsilon \qquad \forall\, n < A$$

$$\Rightarrow \lim\limits_{n\to\infty} a_n = 0$$

35

37

Theorem

Of $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a null sequence, Then $\{a_n b_n\}$ is a null sequence.

Proof #

∵ $\{b_n\}$ is a bounded sequence

∴ ∃ a real number M such that

$$|b_n| \leq M \quad \forall n.$$

Also $\{a_n\}$ is a null sequence

⟹ Given $\varepsilon > 0$, ∃ a the integer m such that

$$|a_n| < \frac{\varepsilon}{M} \quad \forall n \geq m.$$

$$|a_n b_n - 0| \geq |a_n| |b_n|$$

$$< \frac{\varepsilon}{M} \cdot M = \varepsilon \quad \forall n \geq m$$

⟹ $\lim\limits_{n \to \infty} a_n b_n = 0$

Theorem

Of a sequence $\{a_n\}$ oscillates finitely and $\lim\limits_{n \to \infty} b_n = 0$, then $\lim\limits_{n \to \infty} (a_n b_n) = 0$

Proof #

∵ $\{a_n\}$ oscillates finitely

∴ $\{a_n\}$ is bounded.

By above theorem. $\lim\limits_{n \to \infty} a_n b_n = 0$

Theorem

Of $\{a_n\}$ is a null sequence and c is a Constant, then $\{c a_n\}$ is a null sequence.

Proof

36

$\therefore \{a_n\}$ is a null sequence

$\Rightarrow$ Given $\epsilon > 0$, $\exists$ a +ve integer m.

Such that $|a_n| < \dfrac{\epsilon}{|c|+1}$ $\forall n \geqslant m$

Now

$|ca_n| = |c||a_n| < \left(\dfrac{|c|}{|c|+1}\right) \epsilon < \epsilon$ $\forall n \geqslant m$

$\Rightarrow \lim\limits_{n \to \infty} ca_n = 0$

$\Rightarrow \{ca_n\}$ is a null sequence.

Subsequence

If a new sequence is constructed from the terms of old sequence by picking out terms in any way (but preserving the original order) but in the same order as in original, then new sequence is called a subsequence of the old sequence.

OR

Let $f: N \to N$ be a strictly increasing function with $f(k)$ denoted by n_k. If $\{a_n\}$ is any sequence, then $\{a_{n_k}\}_{k=1}^{\infty} = \{a_{n_k}\} = \{a_n\}$ is called a subsequence of $\{a_n\}$.

Note:
$a: N \to R$ $f: N \to N$

$(a \circ f)(k) = a_{f(k)} = a_{n_k}$

Explanation: 37

Consider the sequence. $\{a_n\}$ defined by

$a_n = \dfrac{1}{n}$ i.e

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \text{---} \longrightarrow \text{①}$$

If we construct a subsequence by crossing out every other term, we get a subsequence.

$$1, \tfrac{1}{3}, \tfrac{1}{5}, \tfrac{1}{5}, \text{----}$$

Sub-sequence index N	f	Super sequence index N	a --- term.
k		n_k	
1	$f \longrightarrow$	$f(k) = n_k.$	
	$f \longrightarrow$	$f(1) = n_1 = 1$	$\longrightarrow a_1 = 1$
2	$f \longrightarrow$	$f(2) = n_2 = 3$	$\longrightarrow a_3 = \tfrac{1}{3}$
3	$f \longrightarrow$	$f(3) = n_3 = 5$	$\longrightarrow a_5 = \tfrac{1}{5}$

---			
k	$f \longrightarrow$	$f(k) = n_k = 2k-1$	$\longrightarrow a_{2k-1}$

$$\dfrac{1}{2k-1} \; _{2}\dfrac{1}{2k-1}$$

Thus subsequence $\{a_{n_k}\} = \{a_{2k-1}\} = \left\{\dfrac{1}{2k-1}\right\}_{k=1}^{\infty}$

a subsequence is $\left\{\dfrac{1}{2n-1}\right\}_{n=1}^{\infty}$

__Note__ (1) we note that n_k is no. of index which

Corresponds to k th term of the subsequence.

(2) $n_k \geqslant k$.

If we cross-out every odd numbered term we get the subsequence.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6} - - - - - -$$

Sub-sequence index → (n_k) sequence index → a_{n_k} a sub-sequence term.

$1 \longrightarrow n_1 = 2 \qquad a \longrightarrow a_2 = \frac{1}{2}$

$2 \longrightarrow n_2 = 4 \qquad a \longrightarrow a_4 = \frac{1}{4}$

$3 \longrightarrow n_3 = 6 \qquad a \longrightarrow a_6 = \frac{1}{6}$

$- \quad - - \qquad |$

$- \quad - - \qquad |$

$k \longrightarrow n_k = 2k \qquad a \longrightarrow a_{2k} = \frac{1}{2k}$

Thus the subsequence is $\left\{\frac{1}{2k}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$

Examples.

The subsequences of the sequence of even integers $\{n\}$ are.

(a) The sub-sequence of even integers $= \{2n\}$
$= \{2, 4, 6, \cdots \}$

(b) The sub-sequence of odd integers $= \{2n-1\}_{n=1}^{\infty}$
$= \{1, 3, 5, - - - \}$

(c) The subsequence of primes $2, 3, 5, 7, 11 - - -$

Remarks # (1) * The terms of a subsequence occur in the same order in which they occur in the original sequence.

(2) # Every sequence is a subsequence of itself

(3) # The interval in its various terms of a subsequence need not be regular.

(4) # Given a term a_m of a sequence $\{a_n\}$, there is a term of the subsequence following it.

Theorem # If a sequence $\{a_n\}$ converges to A, then every subsequence of $\{a_n\}$ converges to A.

Proof # Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$

$\therefore \{a_n\}$ converges to A

$\therefore \{a_n\}$ converges to A

Given $\varepsilon > 0, \exists$ a natural no N_1 such that

$$|a_n - A| < \varepsilon \qquad \forall \ n \geqslant N_1 \longrightarrow ①$$

Since n_k is strictly increasing sequence and $n_k \geqslant N_1$

$n_k \ \ n_n \geqslant n \geqslant N_1$. Thus if $k \geqslant N_1$, then

therefore $n_k \geqslant k \geqslant N_1$ and from ①

we have $|a_{n_k} - A| < \varepsilon \qquad \forall \ n_k \geqslant N_1$

$$|a_{n_k} - A| < \varepsilon \qquad \forall \ n_k \geqslant N_1$$

$\Rightarrow \{a_{n_k}\}$ converges to A.

Q.# Let a_n ...

Note # **[1]#** The converse of the above theorem is not true i.e if a subsequence or even an infinitely many subsequences of a given sequence converge, the original sequence may not converge. e.g let $a_n = (-1)^n$. Then $\{a_n\}$ does not converge. However the two subsequences $\{a_{2n}\}$ and $\{a_{2n-1}\}$ Converge to -1 and 1 respectively

[2]# If all subsequences of a sequence $\{a_n\}$ Converge to the same limit, only then $\{a_n\}$ Converges to that limit.

[3]# To prove that a sequence is not convergent it is sufficient to show that two of its subsequences converge to different limit.
In fact $\lim_{n\to\infty} a_n = A$ iff every subsequence converges to the same limit

Example

The sequences $\{\frac{1}{2n}\}$ & $\{\frac{1}{n_i}\}$ are subsequences of the convergent sequence $\{\frac{1}{n}\}$ and $\lim_{n\to\infty}\frac{1}{n}=0$.
Thus $\lim_{n\to\infty}\frac{1}{2n} = \lim_{n\to\infty}\frac{1}{n_i} = 0$

Theorem *(cntm)#

A sequence $\{a_n\}$ Converges to a limit A iff the subsequences of even numbered terms and odd numbered terms i.e $\{a_{2n}\}$ & $\{a_{2n-1}\}$ both converge to A. Then for

Proof # Let $\{a_n\}$ Converges to A. Then for

Given $\epsilon > 0$, $\exists$ a natural no N_1 such that

$$|a_n - A| < \epsilon \qquad \forall n \geq N_1.$$

$$\Rightarrow |a_{2n} - A| < \epsilon \qquad \forall (2n) \geq N_1.$$

and $$|a_{2n-1} - A| < \epsilon \qquad \forall (2n-1) \geq N_1.$$

$\Rightarrow$ Subsequences $\{a_{2n}\}$ & $\{a_{2n-1}\}$ converge to A

Converse Let $\{a_{2n}\}$ & $\{a_{2n-1}\}$ both converge to A. then for given $\epsilon > 0$ $\exists$ natural nos N_1 & N_2 such that

$$|a_{2n} - A| < \epsilon \qquad \forall \ 2n \geq N_1$$

and $$|a_{2n-1} - A| < \epsilon \qquad \forall (2n-1) \geq N_2.$$

Let $N_3 = \max(N_1, N_2)$, then

$$|a_{2n} - A| < \epsilon \qquad \forall (2n) \geq N_3$$

and $$|a_{2n-1} - A| < \epsilon \qquad \forall (2n-1) \geq N_3$$

$$|a_n - A| < \epsilon \qquad \forall n \geq N_3$$

$\Rightarrow \{a_n\}$ converges to A.

Examples

The sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{2^3}, \frac{1}{3^3}, \frac{1}{2^4}, \frac{1}{3^4}, - - -$$

Converges to zero since even numbered and odd numbered terms both converge to 0.

Sequence $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, - - - - -$
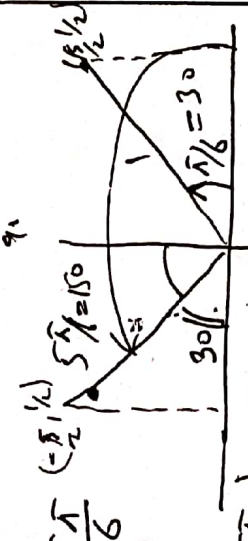
diverges because even and odd numbered terms converge to 0 & 1

Q# Let $a = \frac{54}{43}$ ...

Prove that sequence $\{Sin\, n\}$ diverges

Sol: We use elementary properties of Sine Function.

We note that
$$Sin\, \frac{\pi}{6} = \frac{1}{2} = Sin^{-1} \frac{5\pi}{6}$$

and

$Sin\, x > \frac{1}{2}$ in interval $(\frac{\pi}{6},\ \frac{5\pi}{6}) = I_1$

∴ The length of the interval $I_1 = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3} > 2$

∴ There are at least two natural numbers lying inside I_1. Let m_1 be the 1st such number.

Similarly for each $k \in \mathbb{N}$

$Sin\, x > \frac{1}{2}$ $\forall x \in \left(\frac{\pi}{6}+2\pi(k-1),\ \frac{5\pi}{6}+2\pi(k-1)\right) =$

Let $I_k = \left(\frac{\pi}{6}+2\pi(k-1),\ \frac{5\pi}{6}+2\pi(k-1)\right)$

∴ The length of I_k is greater than 2

∴ There are at least two natural numbers lying inside I_k.

We let m_k be the 1st one. The subsequence $\{Sin\, m_k\}$ of $\{Sin\, n\}$ obtained in this has property that

$Sin\, m_k \in [\frac{1}{2},\ 1]$ $\forall m_k$

Similarly if $k \in \mathbb{N}$ and J_k is the interval

$$J_k = \left(\frac{7\pi}{6}+2\pi(k-1),\ \frac{11\pi}{6}+2\pi(k-1)\right),\ \text{then}$$

$Sin\, x < -\frac{1}{2}$ $\forall x \in J_k.$

42

Theorem # (a) If a sequence $\{a_n\}$ diverges to $+\infty$, then every subsequence of $\{a_n\}$ also diverges to $+\infty$

(b) If a sequence $\{a_n\}$ diverges to $-\infty$, then every subsequence of $\{a_n\}$ also diverges to $-\infty$

Proof # Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$.

∴ $\{a_n\}$ diverges to $+\infty$

∴ For every +ve real no K, however large, $\exists$ a natural no m such that

$$a_n > K \qquad \forall n \geq m.$$

∴ $\{n_k\}$ is strictly increasing sequence

∴ For $k \geq m$, we have $n_k \geq k \geq m$.

and hence for $n_k \geq n_0 \geq m$

$$a_{n_k} > K \qquad \forall n_k \geq m.$$

$\Rightarrow \{a_{n_k}\}$ diverges to $+\infty$

(b) Try yourself

Note Ⅰ# The converse of above theorem is not true i.e if a subsequence of a given sequence diverges to $+\infty$ $(-\infty)$, then the sequence need not diverge to $+\infty$ (or $-\infty$) e.g

let $a_n = (-1)^n n = \begin{cases} -n & \text{if n is odd} \\ n & \text{if n is even} \end{cases}$

Then sub-sequence $\{a_{2n-1}\}$ diverges to $-\infty$ & the sub sequence $\{a_{2n}\}$ diverges to $+\infty$ but the sequence does not diverge to $+\infty$ or $-\infty$ but oscillates

Ⅱ# If all subsequences diverge to $+\infty$ $(-\infty)$ only then the sequence diverges to $+\infty$ $(-\infty)$.

Q# Let a_n be a [sequence] ...

$\because$ length of J_k is greater than 2.

$\therefore$ There are atleast two natural numbers lying inside J_k. Let m_k be the 1st natural number lying in J_k.

The subsequence $\{\sin m_k\}$ of $\{\sin n\}$ is such that

$$\sin m_k \in [-1, -\tfrac{1}{2}]\cup[\tfrac{1}{2}, 1] \quad \forall m_k.$$

Now $[-1, -\tfrac{1}{2}]\cup[\tfrac{1}{2}, 1]$

let c be any real number. Then atleast of the subsequences $\{\sin n_k\}$ & $\{\sin m_k\}$ lie entirely outside $\tfrac{1}{2}$ - nbhd of c ie ic $(c-\tfrac{1}{2}, c+\tfrac{1}{2})$. Therefore c can not be a limit of that subsequence. Since $c \in R$ is an arbitrary, therefore that subsequence is divergent and the sequence is also divergent.

Q# A sequence defined by

$$a_n = \left(1 - \frac{1}{n}\right)\sin\frac{n\pi}{2}$$

diverges

Sol We have.

$$a_{2k} = \left(1 - \frac{1}{2k}\right)\sin k\pi = \left(1 - \frac{1}{2k}\right)\cdot 0 = 0 \qquad k=1,2\ldots$$

$$a_{4k+1} = \left(1 - \frac{1}{4k+1}\right)\sin(4k+1)\frac{\pi}{2}$$

$$= \left(1 - \frac{1}{4k+1}\right)(1) \qquad k=1, 2, \cdots$$

We note that subsequences $\{a_{2k}\}$ and $\{a_{4k+1}\}$ converges to 0 & 1

Convergence of Constant Sequence.

Theorem# A constant sequence is cgt.

$$\text{if } a_n = c \qquad \forall n \in N, \text{ then}$$

$$\text{OR}$$

$$\lim_{n \to \infty} a_n = c.$$

Proof# Let $\epsilon > 0$, then.

$$|a_n - c| = |c - c| = 0 = 0 < \epsilon \quad \forall n \geq 1 = N$$

$$\Rightarrow a_n \to c$$

ALGEBRA OF LIMITS

If $\{a_n\}$ & $\{b_n\}$ are sequences of real nos, we define their sum to be.

$$\{a_n + b_n\} \qquad \text{addition is performed term by term.}$$

Product by $\{a_n b_n\}$

Difference by $\{a_n - b_n\}$

if $b_n \neq \forall n \in N$, then division is defined by

$$\frac{\{a_n\}}{\{b_n\}} = \left\{\frac{a_n}{b_n}\right\}$$

Multiple of sequence $\{a_n\}$ by $c \in R$ is defined by $\{c a_n\}$

Q# Let a be a Constant. Then

__Theorem#__ If $\{a_n\}$ & $\{b_n\}$ be sequence of real nos. that converge to A & B respectively and $c \in R$. Then.

(a) $\lim_{n\to\infty} (a_n+b_n) = A+B$

(a') $\lim_{n\to\infty} ca_n = CA$

(b) $\lim_{n\to\infty} (a_n-b_n) = A-B$

(c) $\lim_{n\to\infty} (a_nb_n) = AB$

(d) If $a_n \neq 0 \ \forall n$ and $A \neq 0$, then

$\lim_{n\to\infty} \dfrac{1}{a_n} = \dfrac{1}{A}$

(e) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ provided $b_n \neq 0$ & $B \neq 0$

__Proof#__ (a₁) $\because \lim_{n\to\infty} a_n = A$

$\therefore$ Given $\varepsilon > 0 \ \exists$ even integer

m such that

$|a_n - A| < \dfrac{\varepsilon}{|c|+1} \qquad \forall n \geq m$

$|Ca_n - CA| = |C||a_n - A|$

$< |C| \dfrac{\varepsilon}{|c|+1} < \varepsilon \qquad \forall n \geq m$ in

$\Rightarrow \lim_{n\to\infty} ca_n = CA$

__OR__

If $c = 0$, then result is obvious because

$|ca_n - CA| = 0 < \varepsilon \qquad \forall n \quad \forall \varepsilon > 0$

Let $c \neq 0$.

$a_n \longrightarrow A$

$\therefore$ For given $\varepsilon > 0 \ \exists$ a natural no m

Such that:

$|a_n - A| < \dfrac{\epsilon}{|c|}$

$|ca_n - cA| = |c||a_n - A|$

$\quad < |c| \dfrac{\epsilon}{|c|} = \epsilon \quad \forall n \geq m$

$\Rightarrow \quad |ca_n - cA| < \epsilon \quad \forall n \geq m$

$\Rightarrow \quad \lim_{n \to \infty} ca_n = cA$.

(b) Given $\epsilon > 0$

$\therefore \quad \lim_{n \to \infty} a_n = A \qquad \lim_{n \to \infty} b_n = B$

$\therefore \ \exists$ natural nos $m_1 \ \& \ m_2$ such that

$|a_n - A| < \epsilon/2 \quad \forall n \geq m_1 \quad \longrightarrow ①$

$|b_n - B| < \epsilon/2 \quad \forall n \geq m_2 \quad \longrightarrow ②$

Let $m = \max(m_1, m_2)$. Then.

$|a_n - A| < \epsilon/2 \qquad \longrightarrow ③$

$|b_n - B| < \epsilon/2 \qquad \longrightarrow ④$

$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)|$

$\qquad \leq |a_n - A| + |b_n - B|$

$\qquad < \epsilon/2 + \epsilon/2 = \epsilon \quad \forall n \geq m$

$\Rightarrow \ \lim_{n \to \infty} (a_n + b_n) = A + B$

Q# Let a be a Constant. Then

(b) $|(a_n+b_n)-(A-B)| = |(a_n-A)-(b_n-B)|$

$\leq |a_n-A|+|b_n-B|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall n \geq N_A, \, n \geq N_B$

$< \varepsilon \quad \forall n \geq N$

$\Rightarrow |(a_n+b_n)-(A-B)| < \varepsilon \quad \forall n \geq N$

$\Rightarrow \lim_{n \to \infty}(a_n+b_n) = A-B$

Note The converse of (a) &(b) not necessarily true i.e existence $\lim_{n \to \infty}(a_n+b_n)$ does not necessarily imply that $\lim a_n$ & $\lim b_n$ also exist. e.g $\}$ let $a_n=n$, $b_n=-n$ both divergent but $a_n+b_n = 0 \quad \forall n$ & $\lim_{n \to \infty}(a_n+b_n)=0$

let $a_n=n$, $b_n=n$

$a_n-b_n = 0$ & $\lim_{n \to \infty}(a_n-b_n)=0$ but $\{a_n\}$ & $\{b_n\}$ are divergent

(c) Let $\varepsilon > 0$ be given.

$|a_n \cdot b_n - AB| = |a_n \cdot b_n - b_n \cdot A + b_n \cdot A - AB|$

$\leq |a_n \cdot b_n - b_n \cdot A| + |b_n A - AB|$

$= |b_n||a_n - A| + |A||b_n - B| \rightarrow ①$

$\because$ The sequence $\{b_n\}$ being cgt is bounded.

$\therefore \exists$ a no M such that

$|b_n| \leq M$

49

$\therefore \lim\limits_{n \to \infty} a_n = A$ & $\lim\limits_{n \to \infty} b_n = B$

$\therefore$ ∃ nos $m_1, m_2 \in \mathbb{N}$ such that

$|a_n - A| < \dfrac{\epsilon}{2M} \quad \forall n \geqslant m_1$

and, $|b_n - B| < \dfrac{\epsilon}{2(|A|+1)} \quad \forall n \geqslant m_2$

$\left[\dfrac{2(|A|+1)}{\text{rather than}} \right.$ used.

$\left. \dfrac{\epsilon}{2|A|} \text{ to avoid} \atop \text{from } A = 0 \right.$

Let $m = \max(m_1, m_2)$. Then,

$|a_n - A| < \dfrac{\epsilon}{2M} \longrightarrow$ ②

$|b_n - B| < \dfrac{\epsilon}{2(|A|+1)} \longrightarrow$ ③

From ① , ② & ③

$|a_n \cdot b_n - AB| \leq |b_n| |a_n - A| + |A| |b_n - B|$

$< M \cdot \dfrac{\epsilon}{2M} + |A| \cdot \dfrac{\epsilon}{2(|A|+1)} \quad \forall n \geqslant m$

$= \dfrac{\epsilon}{2} + \left(\dfrac{|A|}{|A|+1} \right) \dfrac{\epsilon}{2} < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$

$\forall n \geqslant m$ in all A

$\Rightarrow |a_n \cdot b_n - AB| < \epsilon$

$\Rightarrow \lim\limits_{n \to \infty} a_n b_n = AB.$

Q # Let a be a constant. Then

Corollary # Let $\{a_n\}$ be a sequence such that $\lim\limits_{n\to\infty} a_n = l$ and k be any +ve integer. Then $\lim\limits_{n\to\infty} a_n^k = l^k$.

Proof # By induction on k

For $k = 2$

$$\lim\limits_{n\to\infty} a_n^2 = \lim\limits_{n\to\infty} a_n \cdot a_n = l \cdot l = l^2$$

⇒ It is true for $k = 2$

Let it be true for $k = p$ ie

$$\lim\limits_{n\to\infty} a_n^p = l^p \qquad p > 2$$

⇒ $k \cdot \lim a_n^p = l \cdot l^p$

⇒ $\lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} a_n^p = l \cdot l^p$

⇒ $\lim\limits_{n\to\infty} (a_n \cdot a_n^p) = l^{p+1}$

⇒ $\lim\limits_{n\to\infty} a_n^{p+1} = l^{p+1}$

⇒ It is true for $k = p+1$

Thus it is true for all +ve integral values of k

and $\lim\limits_{n\to\infty} a_n^k = l^k$

51

<u>Note</u> the converse of the above theorem.

is not necessarily true i.e existence.

$\lim_{n\to\infty}(a_n b_n)$ does not necessarily imply

that the two limits $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$

also exist. e.g

Let $a_n = b_n = (-1)^n$, then both limits

$\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ do not exist. But

$$a_n b_n = (-1)^n (-1)^n = (-1)^{2n} = 1 \quad \forall n$$

So that $\lim_{n\to\infty}(a_n b_n) = 1$ exists.

(d) Fix $\epsilon > 0$.

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \left|\frac{A - a_n}{a_n A}\right| = \frac{|a_n - A|}{|a_n||A|} \qquad ①$$

$$\lim_{n\to\infty} a_n = A$$

$$\therefore \lim_{n\to\infty} |a_n| = |A|$$

Taking $\epsilon = \frac{|A|}{2}$

$\exists$ a +ve integer m, s. that

$$||a_n| - |A|| < \frac{|A|}{2} \quad \forall n \geq m$$

$$\Rightarrow |A| - \frac{|A|}{2} < |a_n| < |A| + \frac{|A|}{2} \quad \forall n \geq m$$

$$\frac{|A|}{2} < |a_n| < \frac{3|A|}{2}$$

$$|a_n| > \frac{|A|}{2}$$

Q# Let a be a Constant. Then

$$|a_n - A| < \frac{|A|}{2} \qquad \forall\, n > m.$$

Now
$$||a_n| - |A|| \le |a_n - A| < \frac{|A|}{2} \qquad \forall\, n > m$$

$$\Rightarrow -\frac{|A|}{2} \le -|a_n - A| \le |a_n| - |A| \qquad \forall\, n > m$$

$$\le |a_n| - |A|$$

$$\Rightarrow -\frac{|A|}{2} \le |a_n| - |A|$$

$$\Rightarrow \frac{|A|}{2} \le |a_n|$$

$$\Rightarrow \frac{1}{|a_n|} \le \frac{2}{|A|} \qquad \forall\, n > m$$

$$\underline{OR}$$

$$\frac{1}{|a_n|} < \frac{|A|}{2} \qquad \forall\, n > m$$

$$|a_n - A| < \frac{|A|}{2} + |a_n| \qquad \forall\, n > m$$

$$|A| = |A - a_n + a_n| \le |A - a_n| + |a_n| \qquad \xrightarrow{(1)}$$

$$|A| \le |A - a_n| + |a_n| < \frac{|A|}{2} + |a_n| \qquad \forall\, n > m$$

$$\Rightarrow \frac{|A|}{2} < |a_n|$$

$$\Rightarrow \frac{1}{|a_n|} < \frac{2}{|A|} \qquad \forall\, n > m$$

$$\frac{2}{|A|} > \frac{1}{|a_n|} \qquad \forall\, n \geq u \;\; A \geq u$$

$$\frac{1}{|a_n|} < \frac{2}{|A|} \qquad \forall\, n \geq u \;\; A$$

Also $\lim_{n\to\infty} a_n = A$, therefore ∃ natural no.

m_2 s. that $\forall\, n \geq m_2$

$$|a_n - A| < \frac{\epsilon\,|A|^2}{2}$$

Let $m = \max(m_1, m_2)$. Then ∀ n≥u

$$\frac{1}{|a_n|} < \frac{2}{|A|} \qquad \textcircled{3} \quad \forall\, n \geq u\, A$$

$$|a_n - A| < \frac{\epsilon\,|A|^2}{2} \qquad \forall\, n \geq u\, A$$

and $|a_n - A|$

From ① ② & ③

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \frac{|a_n - A|}{|a_n|\,|A|}$$

$$< \frac{2}{|A|}\cdot\frac{1}{|A|}\cdot\frac{\epsilon\,|A|^2}{2} = \epsilon \qquad \forall\, n \geq m.$$

$$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{A}\right| < \epsilon \qquad \forall\, n \geq m.$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{a_n} = \frac{1}{A}$$

__OR__

$$\lim_{n\to\infty} a_n = A$$

For $\epsilon = \dfrac{|A|}{2} > 0 \quad \exists\, m_1 \in N$ s. that

Remarks

terms
essential
non-zero
non-zero
verses
limit of

$\left|\dfrac{A_{bn}}{B}\right| = \left|\dfrac{A}{B}\right|$

$\dfrac{A(B-b_n)}{}$

$+ |A| |b_n$

$\dfrac{|A| |b_n|}{|b_n| |B|}$

$\dfrac{|B|}{2}$

$\dfrac{|B| (A_n}{|b_n|}$

$\dfrac{|B| |a}{|b_n|}$

$\dfrac{|a_n}{|b_n|}$

$\lim_{n \to \infty} b_n$

$|b_n - B|$

$\dfrac{|B|}{2} > $

$|b_n| < $

$|b_n|$

$$\left|\,|b_n| - |B|\,\right| \leq |b_n - B| < \frac{|B|}{2} \qquad \forall\, n \geq n_1$$

$$\Rightarrow |B| - \frac{|B|}{2} < |b_n| < |B| + \frac{|B|}{2} \qquad '' \quad ''$$

$$\Rightarrow |b_n| > \frac{|B|}{2} \qquad \forall\, n \geq n_1$$

$$\Rightarrow \frac{1}{|b_n|} < \frac{2}{|B|} \qquad \forall\, n \geq n_1 \qquad \longrightarrow ②$$

Also $\lim\limits_{n \to \infty} a_n = A$ & $\lim\limits_{n \to \infty} b_n = B$

$\Rightarrow \exists$ natural nos m_2, m_3 such that

$$|a_n - A| < \frac{|B|}{4} \qquad \forall\, n \geq m_2 \longrightarrow ③$$

$$|b_n - B| < \frac{|B|^2}{4(|A|+1)} \qquad \forall\, n \geq m_3 \longrightarrow ④$$

Let $m = \max(m_1, m_2, m_3)$. Then.

$$\frac{1}{|b_n|} < \frac{2}{|B|} \qquad \forall\, n \geq m \longrightarrow ⑤$$

$$|a_n - A| < \frac{|B|}{4} \qquad '' \quad '' \longrightarrow ⑥$$

$$|b_n - B| < \frac{|B|^2}{4(|A|+1)} \qquad '' \quad '' \longrightarrow ⑦$$

From ①, ⑤, ⑥ & ⑦

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| \leq \frac{|a_n - A|}{|b_n|} + \frac{|A|\,|b_n - B|}{|b_n|\,|B|}$$

$$< \frac{2}{|B|} \cdot \frac{|B|}{4} + \frac{2}{|B|} \cdot \frac{|A|}{|B|} \cdot \frac{|B|^2}{4(|A|+1)} \qquad \forall\, n \geq m$$

Q# Let a be a constant.

$$= \frac{55+1}{2} = \frac{56}{2}$$

$$= \frac{\epsilon}{2} + \frac{|A|}{|A|+1} \cdot \frac{\epsilon}{2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall n > m$$

$$\Rightarrow \left| \frac{a_n}{b_n} - \frac{A}{B} \right| < \epsilon \qquad \forall n > m$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$$

Note# The converse of the above theorem is not necessarily true i.e the existence of $\lim \left(\frac{a_n}{b_n}\right)$ does not necessarily imply $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ also exist

two limits $\lim_{n\to\infty} a_n$ & $\lim_{n\to\infty} b_n$ does

e.g $a_n = b_n = n$, then $\lim a_n$, $\lim b_n$

$\frac{a_n}{b_n} = \frac{n}{n} = 1 \Rightarrow 1 \& \lim_{n\to\infty} \frac{a_n}{b_n} = 1$

not exist. But

Theorem# If $\lim_{n\to\infty} a_n = A$ & $A \neq 0$, then

$\exists$ a +ve integer k and natural no m such.

that $|a_n| > k$, $k > 0$ i.e sequence is eventually

bounded away from zero

Proof# $\because A \neq 0$ $\therefore$ $|A| > 0$

Also $\lim_{n\to\infty} a_n = A$

$\Rightarrow \frac{|A|}{2} < |A|$, $\exists$ natural no. m s.t

$\Rightarrow$ For $\epsilon = \frac{|A|}{2}$, $|A| > \frac{|A|}{2}$

$|a_n - |A|$

Q# Let a be a constant. Th...

$$\frac{64}{Sₚ+1 = Sₚ}$$

OR $A > B$ and.

On Contrary let $A > B$ and.

$A - B = 3\epsilon$

So that nbhds

$]B-\epsilon, B+\epsilon[$ & $]A-\epsilon, A+\epsilon[$

of B and A are disjoint

$$\xleftarrow{\quad\underset{A-\epsilon}{}\quad\underset{A+\epsilon}{}\quad\underset{B-\epsilon}{}\quad\underset{B}{}\epsilon\quad\underset{B+\epsilon}{}}{\qquad}$$

$\therefore a_n \to A$ & $b_n \to B$ such that

$\therefore \exists$ natural no m_1 & m_2 such that

$A-\epsilon < a_n < A+\epsilon \qquad \forall\, n \geq m_1$

$B-\epsilon < b_n < B+\epsilon \qquad \forall\, n \geq m_2$

Let $m = \max(m_1, m_2)$

Then $a_n \in]A-\epsilon, A+\epsilon[\qquad \forall\, n \geq m$

$b_n \in]B-\epsilon, B+\epsilon[\qquad \forall\, n \geq m$

$\Rightarrow b_n < a_n \qquad \forall\, n \geq m.$

Which Contradicts the fact that

$a_n \leq b_n \qquad \forall\, n$

Hence our supposition is wrong and.

$A \leq B$

Corollary

If $a_n \geq k$ $\forall\, n$ & $\lim\limits_{n\to\infty} a_n = A$, then $A \geq k$.

Let $b_n = k - a_n$. Then $b_n \leq 0$ $\forall\, n$.

$\Rightarrow \lim b_n \geq 0 \Rightarrow \lim(k - a_n) \geq 0$

$\Rightarrow k - A \geq 0 \Rightarrow k \geq A.$

Theorem # If $\{a_n\}$ be sequence such that

$$a \leq a_n \leq b \quad \forall n \in N$$

Then $a \leq \lim_{n \to \infty} a_n \leq b$

Proof # Consider sequence $\{b, b, b, \cdots\} = \{c_n\}$

Then $a_n \leq c_n \quad \forall n.$

$\Rightarrow \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} c_n$

$\Rightarrow \lim_{n \to \infty} a_n \leq b \longrightarrow ①$

Consider sequence $\{d_n\} = \{a, a, a, \cdots\}$

Then $d_n \leq a_n \quad \forall n.$

$\Rightarrow \lim_{n \to \infty} d_n \leq \lim_{n \to \infty} a_n$

$\Rightarrow a \leq \lim_{n \to \infty} a_n \longrightarrow ②$

By ① & ②

$$a \leq \lim_{n \to \infty} a_n \leq b$$

Theorem # (Squeeze Theorem, Sandwich.
(old squeeze play) Theorem) Trapped sequences

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be the sequences such that

$$a_n \leq b_n \leq c_n \quad \forall n.$$

(1) and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = l \quad \forall n.$

Then $\lim_{n \to \infty} b_n = l.$

60

(ii) For some true integer p

$$a_n \leq b_n \leq c_n \quad \forall \ n \geq p$$

, then

$$\lim_{n \to \infty} b_n = l$$

Proof

Let $\varepsilon > 0$ be given.

$\because \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} c_n = l$

$\therefore \exists$ even integers m_1 & m_2 such that

$$|a_n - l| < \varepsilon \qquad \forall \ n \geq m_1$$

$$|c_n - l| < \varepsilon \qquad \forall \ n \geq m_2$$

Let $m = \max(m_1, m_2)$. Then

$$|a_n - l| < \varepsilon \qquad \forall \ n \geq m$$

$$|c_n - l| < \varepsilon \qquad \forall \ n \geq m$$

$\Rightarrow \quad -\varepsilon < a_n - l < \varepsilon$

$\qquad -\varepsilon < c_n - l < \varepsilon$

Thus $\quad l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon$

$\Rightarrow \quad l - \varepsilon < b_n < l + \varepsilon \qquad \forall \ n \geq m$

$\Rightarrow \quad |b_n - l| < \varepsilon \qquad \forall \ n \geq m$

$$\Rightarrow \quad \lim_{n \to \infty} b_n = l.$$

61

When $a_n \leq b_n \leq c_n$ $\quad \forall n \geq p$

Then we take $m \geq max(m, m', p)$
so that

$|a_n - l| < \epsilon$ $\quad \forall n \leq A$

$|c_n - l| < \epsilon$ $\quad \forall n \geq B$

& $a_n \leq b_n \leq c_n$ $\quad \forall n \geq m$ in

$\Rightarrow l - \epsilon < a_n \leq b_n \leq c_n < l + \epsilon$ $\quad \forall n \geq m.$

$\Rightarrow |b_n - l| < \epsilon$ $\quad \forall n \geq m$

$\Rightarrow \lim_{n \to \infty} b_n = l.$

Applications

Q1 # $\lim_{n \to \infty} \left(\dfrac{\sin n}{n}\right) = 0$

We can not apply $\lim_{n \to \infty} \left(\dfrac{a_n}{b_n}\right) = A/B$
because sequence $\{n\}$ is not convergent
However, we have

$-1 \leq \sin n \leq 1$ $\quad \forall n$

$\Rightarrow \dfrac{-1}{n} \leq \dfrac{\sin n}{n} \leq \dfrac{1}{n}$ $\quad \forall n \geq N$

$\therefore \lim_{n \to \infty} \dfrac{1}{n} = \lim_{n \to \infty} \left(\dfrac{-1}{n}\right) = 0$

$\therefore$ By squeeze theorem.

Q# Let a_n ..

$\underline{63}$

Q# Find $\lim_{n\to\infty} a^{1/n}$, where a is a

Find the real number.

Sol# Case I# of $a \gg 1$, then
$$a^{1/n} \gg 1$$

Let $b_n = a^{1/n} - 1$ where $b_n > 0$
$$a^{1/n} = b_n + 1$$

$\Rightarrow a = (1+b_n)^n \geqslant 1 + n b_n$ Bernoulli's inequality

$\Rightarrow a - 1 \geqslant n b_n$

$\Rightarrow 0 \leq b_n \leq \dfrac{a-1}{n}$

$\therefore \lim_{n\to\infty} \dfrac{a-1}{n} = (a-1)\lim_{n\to\infty}\dfrac{1}{n} = 0$

$\therefore$ By squeeze play

$\lim_{n\to\infty} b_n = 0$

$\Rightarrow \lim_{n\to\infty} a^{1/n} - 1 = 0$

$\Rightarrow \lim_{n\to\infty} a^{1/n} = 1$

$\underline{\text{Case}\#2\#}$ $0 < a < 1$, then $1/a > 1$

$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \dfrac{1}{\left(\frac{1}{a}^{1/n}\right)} = \dfrac{1}{1} = 1$

and

$$\lim_{n \to \infty} \frac{\sin n}{n} = 0$$

62

Q# (a) $\lim_{n \to \infty} \frac{\cos n}{n} = 0$

Sol

$\because \quad -1 \leq \cos n \leq 1$

$\therefore \quad -\frac{1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$

$\Rightarrow$ By squeeze play

$$\lim_{n \to \infty} \frac{\cos n}{n} = 0$$

$0 \leq |\frac{\cos n}{n}| \leq \frac{1}{n}$

$\Rightarrow \lim_{n \to \infty} |\frac{\cos n}{n}| = 0$ by squeeze play

and hence $\lim_{n \to \infty} \frac{\cos n}{n} = 0$

(b) $(-1)^n \frac{1}{n} \to 0$

$\therefore \quad 0 \leq |(-1)^n \frac{1}{n}| \leq \frac{1}{n}$

$\Rightarrow \lim_{n \to \infty} (-1)^n \frac{1}{n} = 0$ by squeeze play

(c) $\frac{\cos^2 n}{3^n} \to 0$ because

$0 \leq \frac{\cos^2 n}{3^n} \leq \frac{1}{3^n}$

$\Rightarrow \lim_{n \to \infty} \frac{\cos^2 n}{3^n} = 0$ by squeeze play

Q# Let a be a Constant. Then. $\underline{64}$

Prove that

(a) $a^n \to 0$ if $0 < a < 1$.

(b) $a^n \to +\infty$ if $a > 1$

Sol# $\therefore \ 0 < a < 1$

$\Rightarrow \dfrac{1}{a} > 1$

Let $\dfrac{1}{a} = 1 + p \qquad p > 0$

$\therefore \ \dfrac{1}{a^n} = (1+p)^n = 1+np+(+ve\ term) \geqslant np$

$\dfrac{1}{a^n} \geqslant np$

$\Rightarrow \ 0 < a^n < \dfrac{1}{np} = \dfrac{1}{p}\cdot\dfrac{1}{n}.$

$\therefore \ \lim\limits_{n \to \infty} \dfrac{1}{p\,n} \to 0$

$\therefore$ By squeeze play $\quad \lim\limits_{n \to \infty} a^n = 0$

(b) If $a > 1$, then

$a = 1 + b \qquad$ where $b > 0$

$a^n = (1+b)^n = 1+nb+(+ve\ terms) > 1+nb+(+ve\ terms) > 1+nb \to \infty$

$\Rightarrow \ a^n \longrightarrow +\infty$ as $n \to \infty$

65

Ratio Test For Convergence

For certain types of sequences, the following result provide quick & easy ratio test for convergence.

Theorem# If $\{a_n\}$ be a sequence such that $a_n \neq 0$, be a sequence of +ve terms and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l \quad \text{where } l<1$$

then $\lim_{n\to\infty} a_n = 0$

Proof#

$\because a_n > 0$

$\therefore \lim_{n\to\infty} a_n > 0 \quad \& \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 0$

$\Rightarrow l > 0$

Let ε be a number such that

$l < \varepsilon < 1$ and let $\epsilon = \varepsilon - l > 0$

Then $\exists$ a natural no m such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon \qquad \forall n \geq m$$

$\Rightarrow \dfrac{a_{n+1}}{a_n} < l + \epsilon = l + \varepsilon - l = \varepsilon \quad \forall n \geq m$ in ——①

$\Rightarrow a_{n+1} < \varepsilon\, a_n \quad \forall n \geq m$ ——②

we putting $n = m,\ m+1 \cdots n-1$

$a_{m+1} < \varepsilon\, a_m$

$a_{m+2} < \varepsilon\, a_{m+1}$

Theorem

Theorem # If $\{a_n\}$ be a sequence such that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l > 1, \text{ then}$$

$$\lim_{n \to \infty} a_n = \infty$$

Proof

∵ $l > 1$ ∴ $l - 1 > 0$

We can choose a +ve number ε such that

$$0 < \varepsilon < l - 1 \quad \text{or} \quad l - \varepsilon > 1$$

∴ $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = l$ ∴ ∃ a even integer

m such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon \qquad \forall\, n \geq m$$

$$\Rightarrow l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon \qquad \forall\, n \geq m$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > l - \varepsilon \quad \Rightarrow \quad \text{\textcircled{A}} \longleftarrow \varepsilon = l - \varepsilon \quad \text{\textcircled{A}}$$

in $1 - u \cdots \cdots u+1, \; u = u$

putting

$$a_{m+1} > \varepsilon\, a_m$$
$$a_{m+2} > \varepsilon\, a_{m+1}$$
$$a_{m+3} > \varepsilon\, a_{m+2}$$
$$\underline{\qquad\qquad\qquad}$$
$$a_{n-2} > \varepsilon\, a_{n-3}$$
$$a_{n-1} > \varepsilon\, a_{n-2}$$
$$a_n > \varepsilon\, a_{n-1}$$

Multiplying all above inequalities

$a_{m+1}\, a_{m+2}\, a_{m+3} \cdots a_{n-2}\, a_{n-1}\, a_n$

$> \dfrac{}{}$ & $a_m\, a_{m+1}\, a_{m+2} \cdots a_{n-3}\, a_{n-2}\, a_{n-1}$

$a_n > \dfrac{n-m}{} \; a_m$

$a_n > \dfrac{a_m}{}$

$|a_n| > \dfrac{|a_m|}{}$

$\because \; \varepsilon > 1$

$\therefore \; \varepsilon^n \to \infty$

$\therefore \; \lim_{n\to\infty} a_n = \infty$

Q# prove that for any real no. $x \neq 0$ $\lim_{n\to\infty} \dfrac{x^n}{n!}$

$\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$

Sol let $a_n = \dfrac{x^n}{n!}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}$

$= \dfrac{x \cdot x^n}{(1+n)\, n!} \cdot \dfrac{n!}{x^n} = \dfrac{x}{1+n}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{x}{1+n}$

$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim_{n\to\infty} \dfrac{x}{1+n} = 0$

$\Rightarrow \; \lim_{n\to\infty} a_n = 0$

Q: Apply the above $\lim_{n\to\infty}$ theorem, where a, b
satisfy $0 < a < 1, b > 1$

Sol # (a) $\{a^n\}$ (b) $\left\{\dfrac{b^n}{2^n}\right\}$

(c) $\left\{\dfrac{b^n}{a^n}\right\}$ (d) $\left\{\dfrac{2^{3n}}{3^{2n}}\right\}$

Sol (a) $a_n = a^n$ $0 < a < 1$

$a_{n+1} = a^{n+1}$

$\dfrac{a_{n+1}}{a_n} = a < 1$

$\Rightarrow \lim_{n\to\infty} a^n \to 0$ $a < 1$

(b) Let $a_n = \dfrac{b^n}{2^n}$

$a_{n+1} = \dfrac{b^{n+1}}{2^{n+1}}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{b^{n+1}}{2^{n+1}} \times \dfrac{2^n}{b^n} = \dfrac{b}{2}$

Q# Discuss the Convergence of following sequences
where a, b are such that $0 < a < 1, b > 1$

(a) $\{n^2 a^n\}$ (b) $\{b^n/n^3\}$

(c) $\left\{\dfrac{b^n}{n!}\right\}$ (d) $\dfrac{n!}{2^n}$

Sol (a) Let $a_n = n^2 a^n$

$a_{n+1} = (n+1)^2 a^{n+1}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{(n+1)^2 a^{n+1}}{n^2 a^n} = \left(\dfrac{n+1}{n}\right)^2 a$

$= \left(1 + \frac{1}{n}\right)^2 a$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 a$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

$$= a < 1 \qquad \text{overall} \quad a$$

it diverges rapidly. So
overall result is
unknown.

(b) Let $a_n = \dfrac{g^n}{n^2}$

$$a_{n+1} = \frac{g^{n+1}}{(n+1)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{g^{n+1}}{(n+1)^2} \times \frac{n^2}{g^n} = \frac{n^2}{(n+1)^2} g$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} g$$

$$= g$$

$$\Rightarrow \{a_n\} \text{ is cgt if } g < 1$$

(c) Let $a_n = \dfrac{g^n}{n}$

$$\frac{a_{n+1}}{a_n} = \frac{g^{n+1}}{n+1} \times \frac{n}{g^n} = \frac{1}{g}\left(1+\frac{1}{n}\right)\frac{g}{1}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{g}{1} = \frac{g}{1+n}$$

$$a_{n+1} = \frac{g}{1+n}$$

$$\lim_{n\to\infty} a_n = 0$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

(d) Let $a_n = \dfrac{n^n}{n!}$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Monotone Sequences

(a) Monotone Increasing or Non-decreasing

A sequence $\{a_n\}$ is called monotone increasing (moving in one direction) if

$$a_n \leq a_{n+1} \qquad \forall n \in N$$

and is called strictly increasing if

$$a_n < a_{n+1} \qquad \forall n \in N$$

(b) Monotone Decreasing or Non-Increasing Sequence

A sequence $\{a_n\}$ is called non-increasing or monotone decreasing if

$$a_n \geq a_{n+1} \qquad \forall n \in N$$

and is called strictly decreasing if

$$a_n > a_{n+1} \qquad \forall n \in N$$

(c) # Monotonic Sequence

A sequence $\{a_n\}$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing

(d) Strictly Monotonic Sequence.

A sequence is said to be strictly monotonic if it is either strictly monotonically increasing or strictly monotonically decreasing

Testing of Monotonicity of a Sequence.

There are are several methods of testing whether a sequence $\{a_n\}$ monotone or not

(a) Difference b/w Successive Terms

Difference   Classification.

$a_{n+1} - a_n > 0$ strictly increasing

$a_{n+1} - a_n < 0$,, ,, decreasing

$a_{n+1} - a_n > 0$ Non-decreasing

$a_{n+1} - a_n < 0$ Non-increasing

(b) By Ratio of Successive Terms #

Ratio $\qquad$ Classification

$\dfrac{a_{n+1}}{a_n} > 1$ $\qquad$ increasing

$\dfrac{a_{n+1}}{a_n} < 1$ $\qquad$ Decreasing

$\dfrac{a_{n+1}}{a_n} \geq 1$ $\qquad$ Non-decreasing

$\dfrac{a_{n+1}}{a_n} \leq 1$ $\qquad$ Non-increasing

(c) If $f(n) > f(n)$ and f is differentiable. Then.

Derivative $\qquad$ Classification

$f'(n) > 0$ $\qquad$ increasing

$f'(n) < 0$ $\qquad$ decreasing

$f'(n) \geq 0$ $\qquad$ Non-decreasing

$f'(n) \leq 0$ $\qquad$ Non-increasing

(d) Induction $\qquad$ Use induction on n.

Remarks # If $\{a_n\}$ is increasing, then it is bounded below by a_1 and will be bounded if it is bounded above. If $\{a_n\}$ is decreasing, it is bounded above by a_1 and will be bounded if bounded below by a_1 and will be bounded if

49
NA

it is bounded below.

(2) Monotocity is very useful because it prevents the terms of a sequence from oscillating.

Eventually Monotone or Ultimately Monotone:

Sequence.

A sequence is eventually or ultimately monotone if ∃ an integer m such that the sequence is monotone for all $n > m$. i·e sequence is monotone from some term onward.

Sequence is monotone from some term onward.

The Completeness Property of R#

bounded property of R.

Every non-empty set of real numbers that has an upper bound (is bounded above) also has a supremum in R. It is also called the least upper bound property of R.

Theorem# A monotone sequence of real nos.
is convergent iff it is bounded. Further.

a) If $\{a_n\}$ is bounded monotone increasing sequence, then it converges to its supremum i·e

$$\lim_{n \to \infty} a_n = \sup\{a_n : n \in N\} \geq \sup a_n$$

(b) If $\{a_n\}$ is monotone bounded below, then

<u>it</u> Converges to its infimum ie.

$$\lim_{n \to \infty} a_n = \inf\{a_n : n \in N\}$$

$$= \inf_n a_n$$

<u>Proof</u>

<u>Necessary Condition #</u>

Let $\{a_n\}$ be monotone Convergent sequence.

Then we have already proved that every cgt sequence is bound

<u>Converse</u>

Conversely let $\{a_n\}$ a bounded monotone sequence.

Then ∃ nos m & M such that

$$m \leq a_n \leq M \quad \forall n.$$

Then $\{a_n\}$ is either increasing or decreasing

(a) Let $\{a_n\}$ be bounded increasing sequence.

Then the range set $S = \{a_n : n \in N\}$ is bounded above and by least upper bound axiom of R

S has $l.u.b$ exists in R.

Let $L = Sup\{a_n : n \in N\}$

Given any $\epsilon > 0$, $L - \epsilon$ is not an upper bound of $\{a_n\}$ and there is at least one term a_m of $\{a_n\}$ greater than $L - \epsilon$

$$L - \epsilon < a_m. \longrightarrow ①$$

otherwise $L - \epsilon$ will be an upper bound.

77

Since $\{a_n\}$ is monotonically increasing
sequence

$$a_m \le a_{m+1} \le a_{m+2} \le ---- \to ②$$

By ① & ②

$$L - \varepsilon < a_n \to ③ \qquad \forall n \in \mathbb{N}$$

Also L is least upper bound of sequence

$$\therefore \quad a_n \le L \to ④ \qquad \forall n \in \mathbb{N}$$

From ③ & ④

$$\Longrightarrow \quad L - a_n \ge 0 \quad \forall n$$

$$0 \le L - a_n < \varepsilon \qquad \forall n \in \mathbb{N}$$

$$\Longrightarrow \quad |L - a_n| < \varepsilon \qquad \forall n \in \mathbb{N}$$

$$\Longrightarrow \quad \lim_{n \to \infty} a_n = L.$$

$\underline{OR}$

Since L is supremum, therefore.

$$a_m \le L < L + \varepsilon \qquad \forall n \in \mathbb{N}$$

$$\Longleftarrow \quad a_n < L + \varepsilon \qquad \forall n \in \mathbb{N}$$

By ③ & ⑤

$$L - \varepsilon < a_n < L + \varepsilon$$

$$\Longleftarrow \quad |a_n - L| < \varepsilon$$

$$\Longrightarrow \quad \lim_{n \to \infty} a_n = L.$$

(b) Suppose the sequence $\{a_n\}$ is bounded monotically decreasing sequence.

$M \geq a_1 \geq a_2 \geq a_3 \geq a_4 \geq \ldots \geq a_m$

Let L_1 be the g.l.b of $S = \{a_n : n \in N\}$

Given any $\varepsilon > 0$, $L_1 + \varepsilon > L_1$ and so $L_1 + \varepsilon$ is not a lower bounds of $\{a_n\}$

$\Rightarrow \exists$ an integer m such that

$a_n \leq a_m < L_1 + \varepsilon \quad \forall n \geq m$

$\Rightarrow a_n < L_1 + \varepsilon \longrightarrow ③$

Also L_1 is g.l.b of $\{a_n\}$

$a_n \geq L_1 > L_1 - \varepsilon \quad \forall n \geq m$

$\Rightarrow a_n > L_1 - \varepsilon \longleftarrow ④$

By ③ & ④ we have

$|a_n - L_1| < \varepsilon \quad \forall n \geq m$

$\Rightarrow \lim\limits_{n \to \infty} a_n = L_1$

or

$L_1 \leq a_n \quad \forall n$

$0 \leq a_n - L_1 \longrightarrow ⑤$

by ③ & ⑤

$|a_n - L_1| > L_1 - \varepsilon \quad \forall n \geq m$

Remarks # The monotonic convergence theorem establishes the convergence of sequence without knowing its limit in advance. It also gives us a way of calculating the limit by evaluating supremum and infimum. Sometimes the supremum and infimum can not found easily but once we know that it exists, it is often possible to evaluate the limit by other methods.

Sequences defined inductly must be treated differently. If such a sequence is known to converge, then value of its limit can sometimes be determined by inductive relation.

Applications

Q # 1 # prove that the sequence defined by

$$a_n = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)$$

is cgt

Sol clearly $a_n > 0$

$$a_{n+1} = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)\left(1 - \frac{1}{(n+1)^2}\right)$$

$$= a_n\left[1 - \frac{1}{(n+1)^2}\right] < a_n$$

⟹ $\{a_n\}$ is decreasing and bounded below

It converges to $l.u.b$ which is ≤ 3

$\therefore \lim_{n \to \infty} a_n \leq 3$

$\therefore a_n \geq 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

≥ 2

$\therefore \lim_{n \to \infty} a_n \geq 2$

Hence $2 \leq \lim_{n \to \infty} a_n \leq 3$

Q#3 Prove that the the the sequence with general term.

$a_n = 2 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

Converges

Sol # $a_n = 2 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!}$

$a_{n+1} = 2 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

$= a_n + \dfrac{1}{(n+1)!} > a_n$

$\Rightarrow \{a_n\}$ is monotone increasing

Now $\dfrac{1}{n!} < \dfrac{1}{2^{n-1}}$

$a_n = 2 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{n!} +$

$< 2 + 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{2^{n-1}}$

Q.2 # Show that the sequence {an}, where

$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

Converges. Also $2 \le \lim a_n \le 3$

Sol $a_{n+1} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!}$

$$= a_n + \frac{1}{(n+1)!} > a_n \quad \forall n.$$

⇒ {an} is monotone increasing. We show that {an} is bounded.

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots \quad n \ge 2 \cdot 2 \cdot 2 \cdots 2 = 2^{n-1}$$

$$⇒ \frac{1}{n!} \le \frac{1}{2^{n-1}}$$

using this we have.

$$a_n = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

$$= 1 + (1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}})$$

$$= 1 + \frac{1 - (\frac{1}{2})^n}{1 - 1/2} = 1 + 2\left[1 - (\frac{1}{2})^n\right]$$

$$< 3 \quad \forall n$$

⇒ {an} increases and is bounded above by 3. So {an} converges.

⇒ {an} is increasing ...

$$< 2 + 1 + \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$

$$= 3 + 2\left[1-\left(\frac{1}{2}\right)^n\right] < 4.$$

Hence $\{a_n\}$ is bounded monotone sequence and so converges.

Q#4 Prove that the sequence $\{a_n\}$ defined by

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is dgt

Sol $a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \frac{1}{n+1}$

$= a_n + \frac{1}{n+1} > a_n$

$\Rightarrow \{a_n\}$ is increasing sequence.

$a_1 = 1$

$a_2 = 1 + \frac{1}{2}$

$a_3 = a_2 + \frac{1}{3} + \frac{1}{4} > \left(\frac{1}{4} + \frac{1}{4}\right) > \frac{1}{2} + a_2$

$a_4 = a_3 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geqslant \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{1}{2} + a_3$

$a_4 > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$+ - - - + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$+ - - - - + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)$$

$$+ - - - + \frac{1}{2^n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2^2 + 1} + \frac{1}{2^2}\right) + \left(\frac{1}{2^2 + 1} + \frac{1}{6} + \frac{1}{7} + \frac{1}{2^3}\right) + \left(\frac{1}{2^3 + 1} + \frac{1}{10} + \cdots - + \frac{1}{2^4}\right)$$

$$+ - - - - - + \frac{1}{2^n}$$

$$+ \left(\frac{1}{2^{n-1} + 1} + - - - - - + \frac{1}{2^n}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{2^2}\right) + \left(\frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^3}\right) + \left(\frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4}\right)$$

$$+ \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} + \cdots + \left(\frac{1}{2^n} + \frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}\right)$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}\right)$$

$$(n-1) \text{ terms}$$

$$= 1 + \frac{n}{2}$$

$$\frac{a^n}{2^n} > 1 + n\frac{8y}{2}$$

⟹ Sequence a_n is unbounded and hence.

⟹ Sequence divergent

Q:5 # Prove that the sequence.

Sol:

$$x_1 = \frac{x_0}{a+x_0}, \quad x_2 = \frac{x_1}{a+x_1}, \quad \cdots, \quad x_n = \frac{x_{n-1}}{a+x_{n-1}}$$

$a>1$, $x_0 > 0$ Converges

$$x_n = \frac{x_{n-1}}{a+x_{n-1}}$$

⟹ $x_n < x_{n-1}$

⟹ Sequence $\{x_n\}$ is decreasing

2ndly Since $a>1$, $x_0>0$, They are all of the terms are +ve which means that sequence is bounded below. Thus the sequence is monotone and bounded. Hence it is cgt.

Q#6 prove that the sequence with general term

$$a_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1} + \cdots + \frac{1}{5^n+1}$$

i.e.

$a_1 = \frac{1}{5+1}$

$a_2 = \frac{1}{5+1} + \frac{1}{5^2+1}$

$a_3 = \frac{1}{5+1} + \frac{1}{5^2+1} + \frac{1}{5^3+1}$ is cgt

Sol

$a_{n+1} = \frac{1}{5+1} + \frac{1}{5^2+1} + \cdots + \frac{1}{5^n+1} + \frac{1}{5^{n+1}+1}$

$= a_n + \frac{1}{5^{n+1}+1}$

$\Rightarrow a_{n+1} > a_n$

$\Rightarrow \{a_n\}$ is increasing

Also since $\frac{1}{5^{n+1}+1} < \frac{1}{5^n}$ $\forall n$.

We have:

$a_n = \frac{1}{5+1} + \frac{1}{5^2+1} + \cdots + \frac{1}{5^n+1}$

$< \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^n}$

$= \frac{\frac{1}{5} - \frac{1}{5^{n+1}}}{1 - \frac{1}{5}} = \frac{5}{4}\left[\frac{1}{5} - \frac{1}{5^{n+1}}\right]$

$= \frac{1}{4}\left[1 - \frac{1}{5^n}\right] < \frac{1}{4} \quad \forall n.$

Hence the sequence is convergent.

Q#7* prove that the sequence

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$

is convergent and its limit lies between

½ and 1

Sol/

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$

$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$

$a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$

$\qquad > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1}$

$\qquad = \frac{2}{2n+2} - \frac{1}{n+1} = 0$

$a_{n+1} > a_n$

$\Rightarrow \{a_n\}$ is monotonically increasing

Also

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$

$\qquad < \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}$

$\qquad = \frac{n}{n+1} < 1$

$\Rightarrow \{a_n\}$ is bounded above by 1

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}$

$\qquad > \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$

$a_n > \frac{1}{2}$

Thus $\frac{1}{2} < a_n < 1 \quad \forall n.$

Hence $\frac{1}{2} \leq \lim_{n \to \infty} a_n \leq 1$

Q:8 $\overset{\text{R.G.B.}}{\underset{\text{Ganhil}}{\#}}$ Let $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$

$\forall n \in N.$ Prove that $\{x_n\}$ is increasing and bounded and hence converges.

Sol We note that

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k} \quad \forall k \geq 2 \longrightarrow ①$$

we will use this fact

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$$

$$x_{n+1} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2}$$

$$= x_n + \frac{1}{(n+1)^2} > x_n \quad \forall n.$$

$\Rightarrow \{x_n\}$ is increasing

Also by using ①

$$x_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$< \frac{1}{1^2} + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$+ \cdots - \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \frac{1}{1^2} + \frac{1}{1} - \frac{1}{n} = 2 - \frac{1}{n}$$

$$\forall n \geq \frac{1}{1} . \frac{1}{n}$$

⟹ {αₙ} is bounded an...
is cgt

Calculation of S...

Q#(9)#(a) A sequence
$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}$$
Prove that {aₙ} converge...

Rₐ·B
(b)# Let $a > 0$, we...
which converges to $\sqrt{a}$
Let $s_1 > 0$ be an ar...
$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$
prove that {sₙ} conver...
Note it is general case of

Sol# (a) $a_1 = 2.$

$a_2 = \frac{1}{2}\left(2\right.$

$a_3 = \frac{1}{2}\left(\frac{3}{2}\right.$

$a_4 = \frac{1}{2}\left(\frac{1}{1}\right.$

Casual readers may...
decreasing.

OR

$\frac{3}{2} = 1.5$

1.4167

$= \frac{577}{408} \; 21.4...$

$t \{a_n\}$ i...

⟹ $\{x_n\}$ is bounded and increasing. hence $\}$
is cgt

$\frac{88}{}$

Calculation of Square root

Q #(a) # (a) A sequence $\{a_n\}$ is defined by

$$a_1 = 2, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

Prove that $\{a_n\}$ Converges and find its limit

P9:B
(b) # Let $a > 0$, we construct a sequence.
which converges to $\sqrt{a}$

Let $\delta_1 > 0$ be an arbitrary and.

$$\delta_{n+1} = \frac{1}{2}\left(\delta_n + \frac{a}{\delta_n}\right) \qquad \forall n \in N$$

prove that $\{\delta_n\}$ Converges and find its limit.

Note δ is general case of (a)

$\underline{Sol}$ # (a) $a_1 = 2.$

$$a_2 = \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2} = 1.5$$

$$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = 1.4167$$

$$a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\frac{17}{12}}\right) = \frac{577}{408} = 1.41142$$

Casual reader may deduce that $\{a_n\}$ is
decreasing.

$\underline{OR}$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

$$a_n^2 = \frac{1}{4}\left\{a_{n-1} + \frac{2}{a_{n-1}}\right\}^2$$

$$= \frac{1}{4}\left[\left(a_{n-1} - \frac{2}{a_{n-1}}\right)^2 + 8\right]$$

$$= \frac{1}{4}\left(a_{n-1} - \frac{2}{a_{n-1}}\right)^2 + 2 \geq 2 \quad \forall n \geq 2.$$

$$\Rightarrow a_n^2 \geq 2.$$

$$\Rightarrow a_n \geq \frac{2}{a_n}. \quad \forall n \geq 2.$$

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \leq \frac{1}{2}(a_n + a_n) = a_n.$$

$$a_{n+1} \leq a_n \quad \forall n \geq 2.$$

$$\Rightarrow \{a_n\} \text{ is ultimately decreasing.}$$

OR.

By $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$

$$2a_{n+1}a_n = a_n^2 + 2.$$

$$\Rightarrow a_n^2 - (2a_{n+1})a_n + 2 = 0$$

$$\Rightarrow a_n \text{ satisfies the quadratic equation.}$$

$$\Rightarrow a_n \quad \therefore \text{ This quadratic equation has}$$

$$a_1 = 2$$

has a real root.

$\Rightarrow$ Disc of it must be non-negative.

$\Rightarrow 4a_{n+1}^2 - 4(2) \geqslant 0 \quad \forall n \geqslant 1$

$\Rightarrow a_{n+1}^2 - 2 \geqslant 0 \quad \forall n \geqslant 1$

$\Rightarrow a_{n+1}^2 \geqslant 2 \quad \forall n \geqslant 1$

$\Rightarrow a_n^2 \geqslant 2 \quad \forall n \geqslant 2$

Now

$a_n - a_{n+1} = a_n - \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$

$\qquad = a_n - \dfrac{1}{2}\left(\dfrac{a_n^2 + 2}{a_n}\right)$

$\qquad = \dfrac{2a_n^2 - a_n^2 - 2}{2a_n}$

$\qquad = \dfrac{1}{2}\dfrac{\left(a_n^2 - 2\right)}{a_n} \geqslant 0 \quad \forall n \geqslant 2 \quad \because a_n^2 \geqslant 2 \ \forall n \geqslant 2$

$\qquad \qquad \qquad \qquad \qquad \qquad a_n > 0$

$\Rightarrow a_{n+1} \leq a_n \quad \forall n \geqslant 2$

$\Rightarrow \{a_n\}$ is decreasing and is bounded.
below by 0

$\Rightarrow \lim\limits_{n \to \infty} a_n$ exists.

$\Rightarrow \lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} a_n = l$.

Let $\lim\limits_{n \to \infty} a_n = l$.

$\cdots$ is +ve root of $x^2 - 2x - 1 = 0$

$$\therefore a_n > 0, \quad a_n > 1$$

$$\therefore l > 0$$

$$\lim_{n\to\infty} a_{n+1} = \frac{1}{2}\left(\lim_{n\to\infty} a_n + \frac{2}{\lim_{n\to\infty} a_n}\right)$$

$$l = \frac{1}{2}\left(l + \frac{2}{l}\right) = \frac{l}{2} + \frac{1}{l}$$

$$2l^2 = l^2 + 2$$

$$l^2 = 2 \implies l = \pm\sqrt{2}$$

$$\because \; l > 0$$

$$\therefore \text{only possibility is } l = \sqrt{2} \approx 1.414214$$

(b) $\quad S_{n+1} = \frac{1}{2}\left(S_n + \frac{9}{S_n}\right)$

$$S_n = \frac{1}{2}\left(S_{n-1} + \frac{9}{S_{n-1}}\right)$$

$$\implies S_n^2 = \frac{1}{4}\left(S_{n-1} + \frac{9}{S_{n-1}}\right)^2$$

$$= \frac{1}{4}\left[\left(S_{n-1} - \frac{9}{S_{n-1}}\right)^2 + 4\cdot 9\right]$$

$$= \frac{1}{4}\left(S_{n-1} - \frac{9}{S_{n-1}}\right)^2 + 9 \qquad \text{Ans}$$

$$\implies S_n^2 \geq 9 \implies S_n \geq 3 \iff \frac{9}{S_n} \leq 3$$

$$\implies S_n \geq \frac{9}{S_n}$$

$$\therefore \ S_n > 0 \quad \forall n$$

$$\therefore \ \ell \geq 0 \quad \forall n$$

$$\lim_{n \to \infty} S_{n+1} = \tfrac{1}{2}\left(S_n + \frac{a}{S_n}\right)$$

$$= \tfrac{1}{2}\left(\lim_{n \to \infty} S_n + \frac{a}{\lim_{n \to \infty} S_n}\right)$$

$$\ell = \tfrac{1}{2}\left(\ell + \frac{a}{\ell}\right)$$

$$2\ell^2 = \ell^2 + a$$

$$\ell^2 = a$$

$$\ell = \sqrt{a}$$

Note

For the purpose of calculation, it is often important to have an estimate of how rapidly the sequence S_n converges to $\sqrt{a}$.

$$\lim_{n \to \infty} S_n = \inf_n S_n = \sqrt{a}$$

$$\therefore \ \sqrt{a} \leq S_n \quad \forall n > 2.$$

$$\therefore \ a \leq S_n^2 \quad \forall n > 2.$$

$$\Rightarrow \ \frac{a}{S_n} \leq S_n \quad \forall n \geq 2$$

$$\Rightarrow \ \frac{a}{S_n} \leq a \qquad \frac{a}{S_n} \leq \frac{a}{\sqrt{a}} \leq S_n.$$

$$0 \leq S_n - \sqrt{a} \leq S_n - \frac{a}{S_n} \qquad \therefore \ \frac{a}{S_n} \leq \sqrt{a}$$

$$= \frac{S_n^2 - a}{S_n} \qquad \forall n \geq 2.$$

$$0 \leq \frac{S_n^2 - a}{S_n}$$

Using this inequality we can calculate $\sqrt{a}$ to any desired degree of accuracy.

Thus $\lim_{n \to \infty} x_n = \ell \geq 2 + \sqrt{}$, is a true root of $x^2 - 2x - 1 = 0$

$$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right) \leq \frac{1}{2}(S_n + S_n) \qquad \forall n \geq 2$$

$$(S_{n+1}) \leq S_n \qquad \forall n \geq 2$$

$$\implies \{S_n\} \text{ is decreasing}$$

$$\underline{OR}$$

$$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right)$$

$$\implies S_n^2 - (2 S_{n+1}) S_n + a = 0$$

$$\implies S_n \text{ satisfies quadratic equation } \& S_n > 0$$

$$\implies \text{This equation has real roots}$$

$$\implies \text{Disc} \geq 0$$

$$4 (S_{n+1})^2 - 4a \geq 0$$

$$(S_{n+1})^2 \geq a. \qquad \forall n \geq 1$$

$$\text{or} \quad S_n^2 \geq a. \qquad \forall n \geq 2$$

$$S_n - S_{n+1} = \frac{1}{2}\left(\frac{S_n^2 - a}{S_n}\right) \geq 0 \qquad \forall n \geq 2$$

$$\implies (S_{n+1}) \leq S_n \qquad \forall n \geq 2 \quad \left[\because S_n^2 \geq a, S_n > 0 \right]$$

$$\implies \{S_n\} \text{ is ultimately decreasing}$$

Also $\qquad S_n \geq 0 \quad \forall n.$

Thus $\{S_n\}$ monotone bounded cond.

hence is cgt. Let $\lim\limits_{n \to \infty} S_n = l$.

Exercise Show that the sequence defined.

by $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$ $n \geq 1$, $a_1 > 0$

Converges to 3.

Q# 10 # G.R.B (example) prove that the sequence defined.

by $x_1 = 2$ $x_{n+1} = 2 + \frac{1}{x_n}$ $\forall n \in \mathbb{N}$

is Convergent and Converges to a +ve root of

equation $x^2 - 2x - 1 = 0$

Sol #

$x_{n+1} = 2 + \frac{1}{x_n} > 2$ $\forall n \geq 1$

& $x_1 = 2$

$\Rightarrow x_n \geq 2$ $\forall n \geq 1$

Also

$x_1 = 2$

$x_2 = 2 + \frac{1}{x_1} = 2 + \frac{1}{2} = \frac{5}{2} = 2.5$

$x_3 = 2 + \frac{1}{x_2} = 2 + \frac{2}{5} = \frac{12}{5} = 2.4$

$x_4 = 2 + \frac{1}{x_3} = 2 + \frac{5}{12} = \frac{29}{12}$

$x_5 = 2 + \frac{12}{25} = 2 + \frac{12}{25} = \frac{70}{25}$

We note that sequence ultimately decrease.

bounded monotone and hence

$\Rightarrow \{x_n\}$ Cgt.

Let $\lim_{n\to\infty} x_n = l$. Then.

$l = 2 + \frac{1}{l} \Rightarrow l^2 - 2l - 1 = 0$

Thus $\lim_{n\to\infty} x_n$ is +ve root of $x^2 - 2x - 1 = 0$

and. $\lim_{n\to\infty} x_n = 1 + \sqrt{2}$

Q#11 (R.G.B) Let $\{y_n\}$ be defined by

$$y_1 = 1 \qquad y_{n+1} = \frac{1}{4}(2y_n + 3) \text{ for } n \geq 1$$

Show that $\lim_{n\to\infty} y_n = \frac{3}{2}$

Sol: By direct calculation.

$$y_2 = \frac{5}{4}$$

Hence $\quad y_1 < y_2 < 2$.

We show by induction that $y_n < 2$ then

It is true for $n = 1, 2$.

Let $\quad y_k < 2 \qquad$ For some $k \in \mathbb{N}$.
$\qquad k > 2$

Then $\quad y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2\cdot 2 + 3) = 7/4 < 2$

$\Rightarrow y_{k+1} < 2 \Rightarrow y_n < 2 \quad \forall n \in \mathbb{N}$

We show by induction that $y_n < y_{n+1}$.

For $n = 1 \qquad y_2 = \frac{5}{4} = 1.25 > 1 = y_1$

$\Rightarrow \quad y_1 < y_2$

Let $\quad y_k < y_{k+1} \qquad$ For some $k \in \mathbb{N}$

Then. $\quad 2y_k < 2y_{k+1}$

$2y_k + 3 < 2y_{k+1} + 3$

$y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{1}{4}(2y_{k+1} + 3) = y_{k+2}$.

Thus $g_n < 2$ $\quad \forall n \geq 2$

Also $g_1 = 1$

$\Rightarrow 1 \leq g_n < 2 \quad \forall n \in N$

$\Rightarrow \{g_m\}$ is bounded.

OR

$g_2 = \sqrt{2} < 2$

$g_3 = \sqrt{2 g_2} < \sqrt{2.2} = 2$

$g_4 = \sqrt{2 g_3} < \sqrt{2.2} = 2$

$g_5 = \sqrt{2 g_4} < \sqrt{2.2} = 2$

$-$

$\Rightarrow g_n < 2 \qquad \forall n \geq 2.$

OR

$g_1 = 1$

$g_2 = \sqrt{2 g_1} = \sqrt{2} = 2^{1/2} \qquad \tfrac{1}{2}$

$g_3 = \sqrt{2 g_2} = \left(2 \cdot \sqrt{2}\right)^{1/2} = 2^{\frac{1}{2}+\frac{1}{4}} \qquad \tfrac{1}{2}+\tfrac{1}{4}$

$g_4 = \sqrt{2 g_3} = \left(2 \cdot 2^{\frac{1}{2}+\frac{1}{2^2}}\right)^{1/2} = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}$

$-$

$g_n = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}}$

$g_n = 2^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}} < 2 \qquad \text{Also}$

$g_{n+1} = 2 \cdot \frac{1}{2^n} \geq g_n$

Thus $y_k < y_{k+1}$

$\Rightarrow y_{k+1} < y_{k+2}$

$\Rightarrow y_n < y_{n+1}$ $\forall n \in N$

$\Rightarrow \{y_n\}$ is increasing and bounded above by 2

$\Rightarrow \{y_n\}$ is cgt.

Let $\lim_{n\to\infty} y_n = l = \lim_{n\to\infty} y_{n+1}$

Then from $y_{n+1} = \frac{1}{4}(2y_n + 3)$

$$l = \frac{1}{4}(2l + 3)$$

$$4l = 2l + 3$$

$$2l = 3 \Rightarrow l = \tfrac{3}{2} = 1.5$$

Q # 12 (R.G.B) Show that the sequence.

of real numbers defined by

$\delta_1 = 1$ $\delta_{n+1} = \sqrt{2\delta_n}$ Converges

and $\lim_{n\to\infty} \delta_n = 2$.

Sol

$\delta_1 = 1$ $\delta_2 = \sqrt{2} < 2$

$\Rightarrow \delta_1 < \delta_2 < 2$ $\delta_2 = \sqrt{2} < 2 \longrightarrow$ ①

For $n = 2$, we have

$\delta_2 = \sqrt{2} < 2$.

Let $\delta_k < 2$ for $k > 2$, $k \in N$.

Then $\delta_{k+1} = \sqrt{2\delta_k} < \sqrt{2.2} = 2$.

Q.12 Show that the sequence $\{a_n\}$ defined by

$$a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2\,a_n}$$

Converges to 2

Sol

$a_1 = \sqrt{2} < 2.$

$a_2 = \sqrt{2a_1} < \sqrt{2 \cdot 2} < 2.$

$a_3 = \sqrt{2a_2} < \sqrt{2 \cdot 2} < 2.$

$- - - - - - -$

$- - - - - - -$

$a_n < 2 \qquad \forall n.$

Also $a_1 = \sqrt{2}$

$a_2 = \sqrt{2\sqrt{2}} > \sqrt{2} = a_1.$

$a_2 > a_1.$

Let $a_{k+1} > a_k \longrightarrow ①$ for some $k \in \mathbb{N}$

Then $a_{k+2} = \sqrt{2\,a_{k+1}}$ & $a_{k+1} = \sqrt{2\,a_k}$

From ① $\quad 2a_{k+1} > 2a_k$

$\Rightarrow \sqrt{2\,a_{k+1}} > \sqrt{2a_k}$

$a_{k+2} > a_{k+1}.$

$a_{k+1} > a_k \Rightarrow a_{k+2} > a_{k+1}$

$\Rightarrow \{a_n\}$ is bounded and increasing.

OR

Increasing fact can be proved by induction as

$$a_1 < a_2 \qquad \rightarrow ①$$

we are to prove that
$$a_n < a_{n+1} \quad \forall n.$$

For $n=1$ inequality is true from ①
Let it be true for $n=k$. i.e.
$$a_k < a_{k+1}. \qquad \rightarrow ②$$

Now $a_{k+1} = \sqrt{2a_k} < \sqrt{2a_{k+1}} = a_{k+2}$. by ②

$\Rightarrow a_{k+1} < a_{k+2}$

Thus $a_k < a_{k+1} \Rightarrow a_{k+1} < a_{k+2}$.

$\Rightarrow a_n < a_{n+1} \quad \forall n.$

$\Rightarrow \{a_n\}$ increasing

$\Rightarrow \{a_n\}$ is cgt.

So $\{a_n\}$ increasing sequence & bounded.

Then. Let $\lim\limits_{n\to\infty}\{a_n\} = l$.

$$a_{n+1} = \sqrt{2a_n}$$

$$\Rightarrow l = \sqrt{2l}.$$

$$l^2 = 2l. \qquad l^2 - 2l = 0$$
$$\Rightarrow 2l. \qquad \Rightarrow l(l-2) = 0$$
$$\Rightarrow l = 0 \qquad l = 2. \quad \text{But } 1 \leq l \leq 2$$

$$\Rightarrow \boxed{l = 2}$$

101

$$= \sqrt{2\sqrt{2}} = 2 \cdot 2 \cdot 2 = 2$$

$$= 2^{1/2} \cdot 2^{1/4}$$

$$= \left(2 \cdot 2^{1/2}\right)^{1/2}$$

$$= 2^{\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}}$$

$$= 2$$

$$\frac{1}{2^2} + \cdots + \frac{1}{2^n}$$

$$\frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$\frac{1}{2^{n+1}} \geq a_n \quad \forall n$$

increasing

$$\therefore \ 2^{1/2^n} > 1 \quad \forall n \geq 1$$

$$\therefore \ 2^{\frac{1}{2^{n+1}}} > 1$$

$$\Rightarrow a_n \cdot 2^{\frac{1}{2^{n+1}}} > a_n$$

$$a_{n+1} > a_n$$

$$\forall n.$$

$$< 2 \quad \forall n.$$

ed monotone and verge it. Let $\lim_{n\to\infty} a_n = l$

$a_1 = \sqrt{2}$

$a_2 = \sqrt{2\sqrt{2}}$

$a_3 = \sqrt{2\sqrt{2}}$

$= 2$

$a_n =$

$a_{n+1} =$

$a_{n+1} =$

$\Rightarrow \{a_n\}$

Again $\frac{1}{2} + \cdots$

$n \geq 2$

$\Rightarrow a_n < 2$

$\Rightarrow \{a_n\}$ b

hence

$$\frac{160}{}$$

OR

$a_1 = \sqrt{2}$

$a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}} = \sqrt{2 \cdot 2^{1/2}} = 2^{1/2} \cdot 2^{1/4} = 2^{\frac{1}{2} + \frac{1}{2^2}}$

$a_3 = \sqrt{2a_2} = \left(2 \cdot 2^{\frac{1}{2} + \frac{1}{4}}\right)^{1/2}$

$\qquad = 2^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}$

$\qquad = 2^{\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3}}$

— —

— —

$a_n = 2^{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}}$

$a_{n+1} = 2^{\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}}$

$a_{n+1} = a_n \cdot 2^{\frac{1}{2^{n+1}}} \geq a_n \ \forall n$

$\Rightarrow \{a_n\}$ is increasing

$\boxed{\begin{aligned} 2^{1/n} &> 1 \quad \forall n \\ \therefore \ 2^{\frac{1}{2^{n+1}}} &> 1 \\ \Rightarrow a_n \cdot 2^{\frac{1}{2^{n+1}}} &> a_n \\ a_{n+1} &> a_n \end{aligned}}$

Again. $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} < 2 \ \forall n$

$a_n = 2^{\ } < 2 \ \forall n$

$\Rightarrow a_n < 2 \ \forall n$

$\Rightarrow \{a_n\}$ bounded monotone and hence $\{a_n\}$ is convergent. Let $\lim_{n\to\infty} a_n = l$

$\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$

$\Rightarrow \dfrac{\sqrt{2a_n}}{2a_n^2}$

$\Rightarrow \left(\lim\limits_{n\to\infty} a_n\right) = 2\lim\limits_{n\to\infty}$

$\Rightarrow 2\ell = \ell$

$\ell = 0 \ \&$

but $a_n >$

$\Rightarrow \ell = 2$

$\forall n.$

$\lim\limits_{n\to\infty} a_n$

Show that the sequence

ed by 1

es to 3

$S_2 = \sqrt{3 \cdot S_n}$

$S_3 = \sqrt{3 \cdot 3}$

$S_4 =$

$S_n = 3$

$= \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3}$

$= \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n}$

$= a_n$

$\Rightarrow \dfrac{1}{2} + \cdots + \dfrac{1}{2^n} > a_n$

$\Rightarrow \lfloor \ell =$

$$\sqrt{2} = 1 \Leftarrow$$

Then $\lim_{n\to\infty} a_{n+1} = l$

$a_{n+1} = 2\sqrt{2a_n}$

$\Rightarrow a_{n+1}^2 = 2a_n$

$\Rightarrow (\lim_{n\to\infty} a_{n+1})^2 = 2\lim_{n\to\infty} a_n$

$\Rightarrow l^2 = 2l \Rightarrow l(l-2)=0$

$\Rightarrow l=0 \ \& \ l=2$

But $a_n > 0 \quad \forall n.$

$\Rightarrow l=2 \Rightarrow \lim_{n\to\infty} a_n = 2.$

Q:13 Show that the sequence $\{S_n\}$ defined by $S_{n+1} = \sqrt{3S_n}$, $S_1=1$ converges to 3

Sol

$S_1 = 1$

$S_2 = \sqrt{3S_1} = \sqrt{3} = \sqrt{3}$ \qquad $3^{1/2}$

$S_3 = \sqrt{3S_2} = \sqrt{3\cdot 3^{1/2}} = 3^{\frac{1}{2}+\frac{1}{2^2}}$

$S_4 = 3^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}$

$\underline{\qquad\qquad\qquad}$

$S_n = 3^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}}$

$\Rightarrow S_{n+1} = 3^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}}$

Sol #

$x_n = \sqrt{2+\ldots}$

Suppose

$x = \sqrt{2+1}$

$x > x_1$

$x > 2$

$x > \sqrt{2+\ldots}$

$x > x_{n-1}$ ... induction.

$x_{n+1} > x_n$

Thus $= \sqrt{1} < 2$

$= \sqrt{2+x_1} < \sqrt{2}$

Now $= \sqrt{2+x_2} < \sqrt{2}$

$= \sqrt{3+2} < \sqrt{2}$

$= 2$

OR

$x_1 < 2$

By $x < 2$

Let $x < 4$

Then $x < 2$

$x < 2$

$\Leftarrow$

$< 17 +$

OR $x+2 = \sqrt{2+x_2}$

$\sqrt{2+2}$

A

Sol #

$x_1 = 1$

$\underline{1.3}$

$x_n = \sqrt{2+x_{n-1}}$

$x_2 = \sqrt{2+x_1} = \sqrt{2+1} = \sqrt{3} > 1 = x_1$

$\therefore \; x_2 > x_1$

Suppose that $x_k > x_{k-1}$ for $k \in N$

$\Rightarrow \; 2 + x_k > 2 + x_{k-1}$

$\Rightarrow \; \sqrt{2+x_k} > \sqrt{2+x_{k-1}}$

$\Rightarrow \; x_{k+1} > x_k = x_{k-1}$

Thus $\quad x_{n+1} > x_n \quad \forall n$ by induction.

Now $\quad x_1 = 1 < 2$

$x_2 = \sqrt{2+x_1} < \sqrt{2+2} < 2$

$x_3 = \sqrt{2+x_2} < \sqrt{2+2} < 2$

$x_4 = \sqrt{x_3 + 2} < \sqrt{2+2} < 2$

$\qquad - \qquad - \qquad -$

$x_n < 2 \quad \forall n.$

$\underline{OR} \qquad x_1 < 2$

By induction

Let $\qquad x_k < 2$

Then $\quad 2 + x_k < 4$

$\Rightarrow \quad \sqrt{2+x_k} < 2$

$\Rightarrow \quad x_{k+1} < 2.$

$\underline{OR}$

$x_{k+1} = \sqrt{2+x_k}$

$\sqrt{2+2} = 2$

Hence

$x_n < 2$

$x_k < 2 \qquad 17 + x_{k-1}$

:102

Also $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} < 3$

$S_n = 3$ $\qquad$ $\therefore \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots < 3 \quad \forall\, n$

$\Rightarrow S_n < 3$ $\qquad$ and converges $\forall\, n$.

Hence $\{S_n\}$ is bounded $\qquad$ $S_n > 1$

Let $\lim\limits_{n\to\infty} S_n = l$ $\qquad$ $\therefore \lim\limits_{n\to\infty} S_n > 1$

$\sqrt{3}\,\sqrt{S_n}$

$S_{n+1} = \sqrt{3}\,\sqrt{S_n}$

$S_{n+1}^2 = 3\, S_n$

$\Rightarrow \left(\lim\limits_{n\to\infty} S_{n+1}\right)^2 = 3 \lim\limits_{n\to\infty} S_n$

$\Rightarrow l^2 = 3l$

$l^2 - 3l = 0 \Rightarrow l(l-3) = 0$

$\Rightarrow l = 0 \ \& \ l = 3$

$\therefore \lim\limits_{n\to\infty} S_n > 1 \Rightarrow \lim\limits_{n\to\infty} S_n = 3$

 # 14 If $S_{n+1} = \sqrt{7 S_n}$, $S_1 = 1$, prove that $\{S_n\}$ is convergent. What is its limit.

try yourself.

Thus $\{x_n\}$ bounded $\underset{\text{by}}{}$ monotone sequence.

and $1 \leq a_n < 2$.

$\Rightarrow 1 \leq \lim_{n \to \infty} a_n \leq 2$

Let $\lim_{n \to \infty} x_n = l$.

$x_n = \sqrt{2+x_{n-1}} \Rightarrow x_n^2 = 2+x_{n-1}$

$\Rightarrow (\lim x_n)^2 = 2+ \lim x_{n-1}$

$\Rightarrow l^2 = 2+ l.$

$\Rightarrow l^2 - l - 2 = 0$

$\Rightarrow (l-2)(l+1) = 0$

$\Rightarrow l = 2 \quad l = -1$

$\therefore x_n > 0 \;\forall n \Rightarrow l = 2$

Q.16 Prove that the sequence $\{a_n\}$

defined by $a_1 = \sqrt{7}$ & $a_{n+1} = \sqrt{7+a_n}$

Converges to the true square root of

$x^2 - x - 7 = 0$

Sol

$a_1 = \sqrt{7} \qquad\qquad a_{n+1} = \sqrt{7+a_n}$

$a_2 = \sqrt{7+a_1} = \sqrt{7+\sqrt{7}} > \sqrt{7} = a_1$

Let $a_k > a_{k-1}$ for some $k \in \mathbb{N}$

$7 + a_k > 7 + a_{k-1} \Rightarrow a_{k+1} > a_k.$

$\Rightarrow \sqrt{7+a_k} > \sqrt{7+a_{k-1}}$

By Mathematical induction.

$\Rightarrow \{a_n\}$ is monotonically increasing

$a_{n+1} > a_n. \quad \forall n.$

Now $a_1 = \sqrt{7} < 7$

Let $a_k < 7$ for some $k \in N$

$7 + a_k < 7 + 7 = 14$

$\overline{\sqrt{7+a_k}} < \sqrt{14} < \sqrt{49} = 7$

$\Rightarrow a_{k+1} < 7$

$\Rightarrow$ By mathematical induction.

$a_n < 7 \quad \forall n.$

$\Rightarrow \{a_n\}$ is bounded above

Since $\{a_n\}$ is monotonically increasing and bounded above, it is cgt.

Let $\lim_{n \to \infty} a_n = l$

Now $a_{n+1} = \sqrt{7+a_n} \Rightarrow a_{n+1}^2 = 7+a_n$

$\Rightarrow l^2 = 7+l \Rightarrow l^2 - l - 7 = 0$

$\therefore a_n > 0 \; \forall n.$

$\therefore \{a_n\}$ converges to a +ve root of $x^2-x-7=0$

$l = \dfrac{1 \pm \sqrt{1+28}}{2} = \dfrac{1 \pm \sqrt{29}}{2}$

So $l = \dfrac{1+\sqrt{29}}{2}$

But $\dfrac{1-\sqrt{29}}{2} < 0$

By Mathematical induction.

$$a_{n+1} > a_n \quad \forall n.$$

⇒ {a_n} is monotonically increasing

Now $a_1 = \sqrt{7} < 7$

Let $a_k < 7$ for some $k \in \mathbb{N}$

$$7 + a_k < 7+7 = 14$$

$$\sqrt{7+a_k} < \sqrt{14} < \sqrt{49} = 7$$

⇒ $a_{k+1} < 7$

⇒ By mathematical induction.

$$a_n < 7 \quad \forall n.$$

{a_n} is bounded above

Since {a_n} is monotonically increasing and bounded above, it is cgt.

Let $\lim_{n\to\infty} a_n = l$.

Now $a_{n+1} = \sqrt{7+a_n}$ ⇒ $a_{n+1}^2 = \sqrt{7+a_n}$ ⇒ a_{n+1}^2

$$l^2 = 7+l \Rightarrow l^2 - l - 7 =$$

$a_n > 0$ ∀n

{a_n} converges to a the root l

$$l = \frac{1 \pm \sqrt{1+28}}{2} = \frac{1 \pm \sqrt{29}}{2}$$

$$= \frac{1-\sqrt{29}}{2} < 0 \qquad \text{So } l = \frac{1+\sqrt{29}}{2}$$

Then $x_n < \alpha$

$x_1 > \alpha$, $x_1 > \beta$

$\Rightarrow x_1 > \alpha$

Thus $\{x_n\}$ is increasing or decreasing according

as $x_1 = \sqrt{a}$ is less or greater than the true root of its equation.

Thus $x^2 - x - 9 = 0$ or $x^2 = x + 9$

Thus $x_1 > \alpha \Rightarrow ②$ i.e.

$x_1^2 - x_1 - 9 > 0$

$x_1^2 > x_1 + 9$

$x_1 > \sqrt{x_1 + 9}$

$\sqrt{x_1 + a} < x_1 \Rightarrow x_2 < x_1$

$\Rightarrow \{x_n\}$ is monotonically decreasing

Now $x_n^2 = x_{n-1} + a > x_n + a$ $\therefore x_n < x_{n-1}$

$x_n^2 - x_n - a > 0$ $\forall n$

From ② $\Rightarrow x_n > \alpha$ $\forall n$

$\underline{O12}$

$x_1 > \alpha$.

Let $x_k > \alpha$ for some $k \in N$

$\sqrt{a + x_k} > \sqrt{a + \alpha}$

$\sqrt{a + x_k} > \sqrt{\alpha^2 + \alpha} > \alpha$ $\therefore x_k > \alpha$

$x_{k+1} > \sqrt{\alpha^2 + \alpha} > \alpha$

$x_1 > \alpha$

$x_k^2 > \alpha$

$\alpha > \alpha^2$

$\forall n$ by induction.

Thus $x_n > \alpha$

Q#17 (R.G.B)

of $x_1 = \sqrt{a}$ $(a>0)$ & $x_{n+1} = \sqrt{a+x_n}$ and

$\forall n \in N$ show that $\{x_n\}$ converges and

find its limit

Sol

$$x_{n+1}^2 - x_n^2 = (a+x_n) - (a+x_{n-1})$$
$$= x_n - x_{n-1}$$

Thus we note that $x_{n+1} > x_n$.

$$x_n > x_{n-1} \implies x_{n+1} > x_n.$$
$$\text{and } x_n < x_{n-1} \implies x_{n+1} < x_n.$$

Thus $\{x_n\}$ is a monotone sequence according

Also $\{x_n\}$ is an increasing

as

$$x_2 > x_1 \qquad\qquad a_2 < x_1$$
$$\sqrt{a+x_1} > x_1 \qquad \sqrt{a+x_1} < x_1$$
$$a+x_1 > x_1^2 \qquad a+x_1 < x_1^2$$
$$x_1^2 - x_1 - a < 0 \to ① \qquad x_1^2 - x_1 - a > 0 \to ②$$

$$\implies a + x_1 - a \neq 0 \to ①$$
$$x_1^2 - x_1 - a \neq 0$$

Now product of root β $x_1^2 - x_1 - a$ is one.

$$x_1^2 - x_1 - a \neq 0 \quad -\text{ve}$$

$$= \text{3 roots is} \quad \frac{1-}{-9}$$

and other

$$\implies \text{are } \frac{1 \pm \sqrt{4a+1}}{2} \qquad \alpha_2 \qquad \frac{1+\sqrt{4a+1}}{2} > 0$$

$$x_1 = \frac{1 \pm \sqrt{4a+1}}{2} \qquad -\beta = \frac{(1-\sqrt{4a+1})}{2} > 0$$

$$-\beta = \frac{(1-\sqrt{4a+1})}{2} > 0 \quad B?$$

① $(x_1-\alpha)$ $(x_1+\beta) < 0$ ② $(x_1-\alpha)(x_1+\beta) > 0$

$$\implies (x_1-\alpha) \qquad \therefore x_1 > 0 \qquad (x_1-\alpha) > 0 \implies x_1 +\beta > 0$$

$$\implies x_1 - \alpha < 0 \implies x_1+\beta \neq 0$$

$$x_1 < \alpha_2 = \frac{1+\sqrt{4a+1}}{2}$$

108

Thus $\{x_n\}$ is monotonically decreasing sequence which is bounded below. Let $\lim\limits_{n\to\infty} x_n = l$.

Hence $\lim\limits_{n\to\infty} x_n$ exists. Let $\lim\limits_{n\to\infty} x_n = l$.

We have $l \neq \alpha$.

Now $x_{n+1} = \sqrt{a + x_n}$

$\Rightarrow l^2 = l + a$

$l^2 - l - a = 0$

$l = \dfrac{1 \pm \sqrt{1+4a}}{2} \Rightarrow l = \dfrac{1 + \sqrt{4a+1}}{2} > 0$

from ①

Similarly $x_1 < \alpha$

$\Rightarrow x_1^2 - x_1 - a < 0$

$\Rightarrow x_1 < \sqrt{x_1 + a} = x_2$

$x_1 < x_2$

$\Rightarrow \{x_n\}$ is monotonically increasing

Now $x_n^2 = x_{n-1} + a < x_n + a$.

$x_n^2 - x_n - a < 0$

$\Rightarrow x_n < \alpha$ $\forall n$ by ①.

Thus $\{x_n\}$ is monotonically decreasing sequence & bounded above by α.

Hence its limit exist & $\lim\limits_{n\to\infty} x_n = \dfrac{1 + \sqrt{4a+1}}{2} = \alpha$.

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Q.18 (R.G.B)

Let $a>0$ & let $a_1>0$. Define.

$$a_{n+1} = \sqrt{a+a_n} \qquad \forall n \in N.$$

Show that $\{a_n\}$ is cgt and find the limit.

Sol # $a_{n+1}^2 - a_n^2 = (a+a_n) - (a+a_{n-1})$

$= a_n - a_{n-1}$

$\Rightarrow a_{n+1} - a_n = \dfrac{a_n - a_{n-1}}{a_{n+1} + a_n}$

Since $a_{n+1} + a_n > 0$, it follows that $a_{n+1} - a_n$

and $a_n - a_{n-1}$ have same sign

i.e. $a_{n+1} \gtrless a_n$ iff $a_n \gtrless a_{n-1}$.

$\Rightarrow \{a_n\}$ increasing or decreasing according

as $\{a_n\}$

$\qquad a_2 > a_1$ or $a_2 < a_1$.

Case I of $a_2 > a_1$, then $\{a_n\}$ increasing by

Mathematical induction.

Also

$a_{n+1}^2 - a_n^2 = a_n + b - a_n^2$

$= -(a_n^2 - a_n + b) \longrightarrow ①$

$\therefore a_{n+1} > a_n$.

$a+a_n > a_n^2$

$\Rightarrow a + a_n > a_n^2$

$a_n^2 - a_n - a < 0 \longrightarrow ②$

$\therefore a_n > 0 \qquad \forall n \geq 1$

$\therefore a_n$ is the +ve root of equation.

$a_n^2 - a_n - a = 0$

Let +ve root be α, then $\alpha = \dfrac{1+\sqrt{4a+1}}{2}$

Then other root = $\dfrac{\text{product of roots}}{\alpha}$

$= \dfrac{-a}{\alpha}$

$a_n - a_n - a = (a_n - \alpha)\left(a_n + \dfrac{a}{\alpha}\right)$

from ② $a_n^2 - a_n - a < 0$

$\Rightarrow (a_n - \alpha)\left(a_n + \dfrac{a}{\alpha}\right) < 0$

$a_n + \dfrac{a}{\alpha} > 0$

$\therefore \quad a_n - \alpha < 0$

$\Rightarrow a_n < \alpha \qquad \forall n$

$\Rightarrow \{a_n\}$ is bounded above by the root α

$x^2 - x - a = 0$

Also $0 < a_1 < \alpha$. Then $\{a_n\}$ is increasing

$x^2 - x - a = 0$

Thus $\{a_n\}$ is increasing when a_1 is less than

the root of $x^2 - x - a = 0$ is less than $\alpha = \dfrac{1+\sqrt{4a+1}}{2}$

Thus when $a_1 < \alpha$

Then $\{a_n\}$ bounded and increasing

& hence is cgt

__Case II__ If $a_2 < a_1$, then $\{a_n\}$ is decreasing i.e.

by Mathematical induction.

$$a_{n+1} < a_n.$$

$$\Rightarrow \sqrt{a+a_n} < a_n.$$

$$\Rightarrow a+a_n < a_n^2$$

$$\Rightarrow a_n^2 - a_n - a > 0$$

$$\Rightarrow (a_n - \alpha)\left(a_n + \frac{a}{\alpha}\right) > 0$$

$$a_n + \frac{a}{\alpha} > 0$$

$$\therefore \quad a_n - \alpha > 0$$

$$\Rightarrow a_n > \alpha \qquad \forall n.$$

$\Rightarrow \{a_n\}$ is bounded below by $\alpha = \dfrac{1+\sqrt{4a+1}}{2}$

Also $a_1 \geq \alpha$

Thus if $a_1 > \alpha$, then $\{a_n\}$ is decreasing

and bounded and hence cgt.

Let $\lim_{n\to\infty} a_n = l.$

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = \sqrt{\lim_{n\to\infty} a_n + a}$$

$$l = \sqrt{l+a}.$$

$$\Rightarrow \ell^2 = \ell + a$$

$$\Rightarrow \ell^2 - \ell - a = 0$$

$$\Rightarrow \ell = \frac{1 \pm \sqrt{4a+1}}{2}$$

$\therefore a_n > 0 \quad \forall n \qquad \therefore \lim_{n\to\infty} a_n \geqslant 0$

$$\Rightarrow \ell = \frac{1 + \sqrt{4a+1}}{2} = \alpha \quad \text{the root of}$$

equation $x^2 - x - a = 0$

Q: 19 (Gaskill exercise)

Define a sequence by

$a_1 = k$, $(k > 0)$, $a_{n+1} = \sqrt{k + a_n}$, $n \geqslant 1$

Show that $\{a_n\}$ has a limit and find it.

Sol #

$a_n > 0 \quad \forall n$.

$$a_2 = \sqrt{a_1 + k} = \sqrt{k + k} = \sqrt{2k}$$

$$a_{n+1}^2 - a_n^2 = (a_n + k) - (a_{n-1} + k)$$
$$= a_n - a_{n-1}$$

$$\Rightarrow a_{n+1} - a_n = \frac{a_n - a_{n-1}}{a_{n+1} + a_n}$$

$\therefore a_{n+1} + a_n > 0$

$\therefore a_{n+1} - a_n$ & $a_n - a_{n-1}$ have same sign

$$\underline{113}$$

i.e. $a_{n+1} \gtrless a_n$ iff $a_n \gtrless a_{n-1}$

$\Rightarrow \{a_n\}$ increasing or decreasing according as

$a_2 \gtrless a_1$

$a_2 > a_1 \qquad a_2 < a_1$

<u>Case I</u> If $a_2 > a_1$, then by mathematical induction it can be proved that $\{a_n\}$ is increasing

i.e.

$$\Rightarrow \frac{a_{n+1}}{k + a_n} > a_n$$

$$\Rightarrow \frac{k + a_n}{k + a_n} > a_n$$

$$\Rightarrow a_n^2 - a_n - k < 0 \qquad a_n^2 - a_n - k > 0 \longrightarrow \textcircled{1}$$

$$\forall n. \quad \text{& other } -ve.$$

$\because a_n > 0$

$\therefore$ has one root $+ve$. & other $-ve$.

$\therefore a_n$ is the root of

$$a_n^2 - a_n - k = 0$$

Let the root be $\alpha = \dfrac{1 + \sqrt{4k + 1}}{2}.$

other root $= -\dfrac{k}{\alpha}$

$$\Rightarrow a_n^2 - a_n - k = (a_n - \alpha)\left(a_n + \frac{k}{\alpha}\right)$$

① ⟹ $(a_n - \alpha)\left(a_n + \dfrac{k}{\alpha}\right) < 0$

$a_n + \dfrac{k}{\alpha} > 0$

∴ ⟹ $a_n - \alpha < 0 \Rightarrow a_n < \alpha \quad \forall n.$

Also $a_1 < \alpha$

⟹ $\{a_n\}$ is increasing if $a_1 < \alpha$ &

$a_n < \alpha \quad \forall n.$ ie bounded above by

α, the root of $x^2 - x - k = 0$

⟹ $\{a_n\}$ is monotonic bounded.

__Case II__ of $a_2 < a_1$, then by mathematical

induction $\{a_n\}$ is decreasing i.e.

$a_{n+1} < a_n$

$\sqrt{k + a_n} < a_n$

$k + a_n < a_n^2$

$a_n^2 - a_n - k > 0$

⟹ $(a_n - \alpha)\left(a_n + \dfrac{k}{\alpha}\right) > 0$

$a_n + \dfrac{k}{\alpha} > 0$

∴ $a_n - \alpha > 0 \Rightarrow a_n > \alpha, \quad \forall n.$

⟹ $\{a_n\}$ is decreasing if $a_1 < \alpha$ &

is bounded below by α

Thus $\{a_n\}$ is monotonic bounded.

Sequence and hence cgt.

$\therefore a_n > 0 \;\forall n$

$\therefore \lim_{n\to\infty} a_n \geq 0$

Let $\lim_{n\to\infty} a_n = l$.

$$a_{n+1} = \sqrt{k + a_n}$$

$$a_{n+1}^2 = k + a_n.$$

$\Rightarrow \left(\lim_{n\to\infty} a_{n+1}\right)^2 = k + \lim_{n\to\infty} a_n.$

$\Rightarrow l^2 = k + l.$

$$l^2 - l - k = 0$$

$\Rightarrow l = \dfrac{1 \pm \sqrt{1+4k}}{2}$

$\therefore l \geq 0 \quad \therefore l = \dfrac{1 + \sqrt{1+4k}}{2},$

the root of equation $x^2 - x - k = 0$

Q# 20 (R.4.B)

Let $x_1 > 1$ and $x_{n+1} = 2 - \dfrac{1}{x_n}$, then

Show that $\{x_n\}$ is bounded & monotone. Find the limit

Sol

$x_{n+1} = 2 - \frac{1}{x_n}$, $x_1 > 1$ ∴ $\frac{1}{x_1} < 1$

$x_2 = 2 - \frac{1}{x_1} > 1$ for some $k \in N$

Let $x_k > 1$

$\Rightarrow \frac{1}{x_k} < 1$

$\Rightarrow -\frac{1}{x_k} > -1$

$\Rightarrow 2 - \frac{1}{x_k} > 1$

$\Rightarrow x_{k+1} > 1$

Thus $x_n > 1 \Rightarrow x_{k+1} > 1$

Thus by M. induction. $\forall n$

$x_n > 1$ $\forall n$

$x_n - x_{n+1} = x_n - \left(2 - \frac{1}{x_n}\right)$

$= \frac{x_n^2 - 2x_n + 1}{x_n}$

$= \frac{(x_n - 1)^2}{x_n} > 0$ ∴ $x_n > x_{n+1}$

$\Rightarrow x_n > x_{n+1}$

$\Rightarrow \{x_n\}$ monotonic bounded and hence convergent

∴ $x_n > 1$ $\forall n$

∴ $\lim_{n\to\infty} x_n = l > 1$

$$\lim_{n \to \infty} x_{n+1} = 2 - \frac{1}{\lim_{n \to \infty} x_n}$$

$$\ell = 2 - \frac{1}{\ell}$$

$$\ell^2 = 2\ell - 1$$

$$\ell^2 - 2\ell + 1 = 0$$

$$(\ell - 1)^2 = 0 \quad \Rightarrow \quad \boxed{\ell = 1}$$

Q.21 Grashit of $x_1 \geq 4$ $\qquad x_{n+1} = 3 - \frac{2}{x_n}$

$$x_2 = 3 - \frac{2}{x_1} = 3 - \frac{2}{4}$$

$$= 3 - \frac{1}{2} = \frac{5}{2} = 2.5 > 2.$$

$$x_3 = 3 - \frac{2}{x_2} = 3 - \frac{2}{\frac{5}{2}} = 3 - \frac{4}{5} = \frac{11}{5} = 2.2 > 2$$

Let $x_k > 2$

for some $k \in N, \ k \geq 2.$

$$+ \ \frac{1}{x_k} < \frac{1}{2} \qquad\qquad \text{Thus } x_k > 2$$

$$\Rightarrow \frac{2}{x_k} < 1 \qquad\qquad \Rightarrow x_{k+1} > 2$$

$$\Rightarrow -\frac{2}{x_k} > -1 \qquad\qquad \Rightarrow x_n > 2 \quad \forall n \geq 1.$$

$$\Rightarrow 3 - \frac{2}{x_k} > 2$$

$$\Rightarrow x_{k+1} > 2$$

$$x_n - x_{n+1} = x_n - \left(3 - \frac{2}{x_n}\right)$$

$$= x_n - 3 + \frac{2}{x_n}$$

$$= \frac{x_n^2 - 3x_n + 2}{x_n} > 0$$

$$= \frac{(x_n - 2)(x_n - 1)}{x_n \cdot x_n - 2} > 0 \qquad x_n > 0$$

$$\therefore x_n > 2$$

$$\Rightarrow x_n > x_{n+1}$$

$$\Rightarrow \{x_n\} \text{ is decreasing } \& \text{ bounded and hence}$$

Convergent $\therefore \lim_{n\to\infty} x_n = l$.

Let $\lim_{n\to\infty} x_n = l$.

$$x_{n+1} = 3 - \frac{2}{x_n}$$

$$\Rightarrow \lim_{n\to\infty} x_{n+1} = 3 - \frac{2}{\lim_{n\to\infty} x_n}$$

$$l = 3 - \frac{2}{l}$$

$$l^2 = 3l - 2$$

$$l^2 - 3l - 2 = 0$$

$$(l-2)(l-1) = 0$$

$$\Rightarrow l = 2, \text{ or } l = 1$$

$$\text{But } l \geqslant 2 \Rightarrow l = 2$$

Q.22 (Generalisation of 119 a_2 & a_{21} made myself.)

Define a sequence as

$$a > 2, \quad x_1 > a-1, \quad x_{n+1} = a - \frac{a-1}{x_n} \qquad \rightarrow ①$$

Prove that $\{x_n\}$ converges to a limit. Find the limit.

Sol

$$x_1 > a-1$$

$$x_2 = a - \frac{a-1}{x_1}$$

$$\therefore \quad x_1 > a-1$$

$$1 > \frac{a-1}{x_1}$$

$$\frac{a-1}{x_1} < 1$$

$$-\frac{a-1}{x_1} > -1$$

$$\Rightarrow a - \frac{a-1}{x_1} > a-1$$

$$\Rightarrow x_2 > a-1$$

Let $x_k > a-1$ for some $k > 2$.

$$\Rightarrow \frac{1}{x_k} < \frac{1}{a-1}$$

$$\Rightarrow \frac{a-1}{x_k} < 1$$

$$\Rightarrow a - \frac{a-1}{x_k} > a-1$$

(20)

$\Rightarrow a - \dfrac{a-1}{x_n} > a-1$

$\Rightarrow x_{n+1} > a-1$

Hence $x_n > a-1 \Rightarrow x_{n+1} > a-1$

Thus $x_n > a-1 \quad \forall\, n.$

$x_n - x_{n+1} = x_n - \left(a - \dfrac{a-1}{x_n}\right)$

$= \dfrac{x_n^2 - ax_n + (a-1)}{x_n}$

$= \dfrac{x_n^2 - (a-1)x_n - x_n + (a-1)}{x_n}$

$= \dfrac{x_n\big(x_n - (a-1)\big) - 1\big(x_n - (a-1)\big)}{x_n}$

$\Rightarrow \dfrac{(x_n - 1)\big(x_n - (a-1)\big)}{x_n} \geq 0$

$\therefore x_n > a-1 > 1 \quad \because a > 2$

$\Rightarrow x_n \geq x_{n+1}$

$\Rightarrow \{x_n\}$ is monotone decreasing and bounded & hence convergent

Also $x_n > a-1$

$\Rightarrow \lim_{n\to\infty} x_n \geq a-1$

$l \geq$

Let $\lim\limits_{n\to\infty} x_n = l$

$\Rightarrow$ Now $x_{n+1} = a - \dfrac{a-1}{x_n}$

$$\lim_{n\to 0} x_{n+1} = a - \dfrac{a-1}{\lim\limits_{n\to\infty} x_n}$$

$$l = a - \dfrac{a-1}{l}$$

$$l^2 = la - (a-1)$$

$$l^2 = la + a - 1 = 0$$

$$l^2 - (a-1)l - l + a-1 = 0$$

$$l[\,l - (a-1)\,] - 1[\,l-(a-1)\,] = 0$$

$$\Rightarrow [\,l - (a-1)\,][\,l-1\,] = 0$$

$$\Rightarrow l - (a-1) = 0 \quad \Big| \quad l-1 = 0$$

$$\Rightarrow l = a-1 \quad \Big| \quad l = 1$$

$$\Rightarrow l \geq a-1 \quad \Rightarrow l = a-1.$$

But $l \geq a-1$.

Q# 23 (R.G.13 exercise 3.3)

Let $x_1 \geq 2$ & $x_{n+1} = 1 + \sqrt{x_n - 1}$, $\forall n \in N$

Show that $\{x_n\}$ is decreasing and bounded.
below by 2. Find the limit.

So! $x_1 \geq 2$

$$x_1 \geq 2 \Rightarrow x_1 - 1 \geq 1$$

$$x_2 = 1 + \sqrt{x_1 - 1} \geq 2 \qquad \therefore x_1 \geq 2$$

$$\underline{122}$$

Let $x_k \geq 2$ for some $k \in N$

$\Rightarrow x_k - 1 \geq 1$

$\Rightarrow \sqrt{x_n - 1} \geq 1$

$\Rightarrow 1 + \sqrt{x_{n-1}} \geq 2$

$\Rightarrow x_{n+1} \geq 2$.

Thus $x_n \geq 2 \quad \forall n$.

$x_n - x_{n+1} = x_n - 1 - \sqrt{x_n - 1}$

$$= \left(\sqrt{x_n - 1}\right)^2 - \sqrt{x_n - 1}$$

$$= \sqrt{x_n - 1}\left(\sqrt{x_n - 1} - 1\right) \geq 0 \quad \because x_n \geq 2.$$

$\Rightarrow x_n \geq x_{n+1}$

$\Rightarrow \{x_n\}$ is monotone decreasing & bounded.
and hence convergent.

Also $x_n \geq 2$

$\Rightarrow \lim_{n \to \infty} x_n \geq 2$.

Let $\lim_{n \to \infty} x_n = \ell$.

$x_{n+1} = 1 + \sqrt{x_n - 1}$

$\lim_{n \to \infty} x_{n+1} = 1 + \sqrt{\lim_{n \to \infty} x_n - 1}$

$\ell = 1 + \sqrt{\ell - 1}$

$\Rightarrow l-1 = \sqrt{l-1}$

$(l-1)^2 = l-1$

$\Rightarrow (l-1)^2 - (l-1) = 0$

$\Rightarrow (l-1)[l-1-1] = 0$

$\Rightarrow (l-1)(l-2) = 0$

$(l \neq 1)\quad l=2$

Thus $l=2$ $\quad\therefore l \neq 1^2$

$\Rightarrow \lim_{n\to\infty} a_n = 2.$

Q#24 (Generalisation of Q23 made by myself.)

Let $x_1 \geqslant a^2+1$, $a \geqslant 1$ & $x_{n+1} = a^2 + \sqrt{x_n - a^2}$ $\forall n \in N$. Show that $\{x_n\}$ decreasing and. bounded below a^2+1. Find the limit.

Sol

$x_1 \geqslant a^2+1.$

$x_2 = a^2 + \sqrt{x_1 - a^2} \geqslant a^2+1 \quad \because x_1 \geqslant a^2+1$

Let $x_k \geqslant a^2+1$ For som $k \in N$

$\Rightarrow x_k - a^2 \geqslant 1 \quad \Rightarrow \sqrt{x_k - a^2} \geqslant 1$

$\Rightarrow a^2 + \sqrt{x_k - a^2} \geqslant a^2+1.$

$\Rightarrow x_{k+1} \geqslant a^2+1$

Thus $x_n \geqslant a^2+1$

124

$$x_n - x_{n+1} = x_n - \left(a^2 + \sqrt{x_n - a^2}\right)$$

$$= x_n - a^2 + \sqrt{x_n - a^2}$$

$$= \sqrt{x_n - a^2}\left(\sqrt{x_n - a^2} - 1\right) \geq 0 \quad ; \quad x_n \geq a^2 + 1$$

$\Rightarrow x_n \geq x_{n+1}$

$\Rightarrow \{x_n\}$ is monotone decreasing and bounded.

and hence convergent $a^2 + 1$.

$\therefore x_n \geq a^2 + 1$

$\therefore \lim_{n \to \infty} x_n \geq a^2 + 1$

Let $\lim_{n \to \infty} x_n = l$.

$$x_{n+1} = a^2 + \sqrt{x_n - a^2}$$

$$\lim_{n \to \infty} x_{n+1} = a^2 + \sqrt{\lim_{n \to \infty} x_n - a^2}$$

$$l = a^2 + \sqrt{l - a^2}$$

$$l - a^2 = \sqrt{l - a^2}$$

$$(l - a^2)^2 = l - a^2$$

$$(l - a^2)^2 - (l - a^2) = 0$$

$$(l - a^2)\left[l - a^2 - 1\right] = 0$$

$$\Rightarrow l = a^2 \quad \text{or} \quad l = a^2 + 1$$

$$\Rightarrow l = a^2 + 1$$

Thus $\lim_{n \to \infty} x_n = a^2 + 1$

$$\underline{\underline{Q.25}} \quad \text{A sequence } \{S_n\} \text{ is}$$

defined as

$$S_1 = a > 0 \qquad S_{n+1} = \sqrt{\dfrac{ab^2 + S_n^2}{a+1}}$$

$$b > a \qquad n \geqslant 1$$

Show that $\{S_n\}$ is cvgt and find its limit

$\underline{\underline{Sol}}$

It is given that

$$a < b$$
$$\Rightarrow S_1 < b \qquad \longrightarrow \text{①}$$

$$S_{n+1}^2 - b^2 = \dfrac{ab^2 + S_n^2}{a+1} - b^2$$

$$= \dfrac{S_n^2 - b^2}{a+1} < 0 \quad \text{when even } S_n^2 - b^2 < 0$$

$$\text{when } S_n^2 - b^2 < 0$$

$$\Rightarrow S_{n+1}^2 - b^2 < 0$$

$$\Rightarrow S_{n+1} < b \qquad \text{when } S_n < b \quad \longrightarrow \text{②}$$

By ① & ②

By mathematical induction, we have

$$S_n < b \quad \forall n$$

$$\underline{OR}$$

$$S_1 < b \Rightarrow S_n < b \quad \text{for } n = 1$$

Let $S_k < b \quad$ for $k \geqslant 1$

$$\Rightarrow S_k^2 - b^2 < 0$$

$$\Rightarrow S_k^2 < b^2$$

$$\underline{126}$$

$\Rightarrow ab^2 + S_k^2 < b^2 + ab^2$

$\Rightarrow \dfrac{ab^2 + S_k^2}{a+1} < \dfrac{b^2(a+1)}{a+1} \qquad \because a+1 > 0$

$\Rightarrow \sqrt{\dfrac{ab^2 + S_k^2}{a+1}} < b$

$\Rightarrow S_{k+1} < b$

Thus by mathematical induction

$$S_n < b \quad \forall\, n$$

$\Rightarrow \{S_n\}$ is bounded above.

Again

$S_{n+1}^2 - S_n^2 = \dfrac{ab^2 + S_n^2}{a+1} - S_n^2$

$\qquad\qquad = \dfrac{a\left(b^2 - S_n^2\right)}{a+1} > 0 \qquad [\because S_n < b]\ \forall\, n$

$\Rightarrow S_{n+1}^2 > S_n^2$

$\Rightarrow |S_{n+1}| > |S_n|$

$\Rightarrow |S_{n+1}| > |S_n|$

$\Rightarrow S_{n+1} > S_n \qquad [\because S_n > 0]$

$\Rightarrow \{S_n\}$ is increasing

Thus $\{S_n\}$ is monotonically increasing and bounded above

note $|a| > |b| \Rightarrow a > b$
but if $a > 0,\ b > 0$, then
$|a| > |b| \Rightarrow a > b$
Also $|a| < |b| \Rightarrow a < b$
if $a > 0,\ b > 0$, then
$a > b \Rightarrow |a| > |b|$

127

and hence convergent.

Let $\lim\limits_{n\to\infty} s_n = l$

Now $s_{n+1}^2 = \dfrac{ab^2 + s_n^2}{a+1}$

$\Rightarrow \left(\lim\limits_{n\to\infty} s_{n+1}\right)^2 = \dfrac{ab^2 + \left(\lim\limits_{n\to\infty} s_n\right)^2}{a+1}$

$\Rightarrow l^2 = \dfrac{ab^2 + l^2}{a+1}$

$al^2 + l^2 = ab^2 + l^2$

$\Rightarrow l^2 = b^2 \Rightarrow l = \pm b$

$\because s_n > 0 \quad \forall n.$

$\therefore \lim\limits_{n\to\infty} s_n > 0 \quad \forall n.$

Hence sequence $\{s_n\}$ converges to b

Q#26 If a_1, b_1 are two +ve unequal numbers and a_n, b_n are defined as

$$a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}) \qquad b_n = \sqrt{a_{n-1}\, b_{n-1}} \qquad n \geq 2.$$

prove that two sequences $\{a_n\}$ and $\{b_n\}$ are monotonic, one increasing and the other decreasing and that they tend to the same limit.

Sol Let $\qquad a_1 > b_1.$

Since for any two +ve numbers, the A.M is greater than the G.M.

$\therefore \quad a_n > b_n.$

Also $\quad a_{n+1} = \tfrac{1}{2}(a_n + b_n) < \tfrac{1}{2}(a_n + a_n) = a_n$
$\therefore \ b_n < a_n.$

$\Rightarrow \{a_n\}$ is $\downarrow$

$\Rightarrow a_1 > a_2 > a_3 > a_4 > a_5 > \cdots$

$\Rightarrow a_1 > a_2 > \cdots = b_n.$ $\quad [a_n > b_n]$

Again. $b_{n+1} = \sqrt{a_n \cdot b_n} > \sqrt{b_n \cdot b_n} = b_n$

$\therefore \{b_n\}$ is monotone decreasing

Now $a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1}) > \tfrac{1}{2}(b_{n-1} + b_{n-1}) = b_{n-1}$

$a_n \geq \tfrac{1}{2}(a_{n-1} + b_{n-1}) > \tfrac{1}{2}(b_{n-1} + b_{n-1}) = b_{n-1}$

$\therefore a_n > b_n$

$a_n > b_{n-1} > b_{n-2} \cdots > b_2 > b_1.$

$\Rightarrow a_n > b_{n-1} > b_{n-2} \cdots$

$\Rightarrow a_n > b_1 \quad \forall n$

$\Rightarrow \{a_n\}$ is $\downarrow$ and bounded. & hence cgt.

Again $b_n = \sqrt{a_{n-1} \cdot b_{n-1}} < \sqrt{a_{n-1} \cdot a_{n-1}} = a_{n-1}$ by (i)

$b_n < a_{n-1} < a_{n-2} < \cdots < a_2 < a_1.$

i.e $\{b_n\}$ is bounded above and being monotone.

$\therefore \{b_n\}$ is convergent increasing

let $\displaystyle\lim_{n \to \infty} a_n = \ell_1$ & $\displaystyle\lim_{n \to \infty} b_n = \ell_2$

Since $a_n = \tfrac{1}{2}(a_{n-1} + b_{n-1})$

$2a_n = a_{n-1} + b_{n-1}$

$\therefore$

$\Rightarrow 2a_{n+1} = \dfrac{129}{a_n + b_n}$

$\Rightarrow \lim\limits_{n\to\infty} 2a_{n+1} = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

$\Rightarrow 2\ell_1 = \ell_1 + \ell_2$

$\Rightarrow \ell_1 = \ell_2$

$\Rightarrow \{a_n\} \,\&\, \{b_n\}$ converge to the same limit.

Q#27 If $a_1 > 0, b_1 > 0$ and $a_n = \sqrt{a_{n-1}\cdot b_{n-1}}$

and $b_n = \dfrac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}}$, prove that

(i) $\{a_n\}$ and $\{b_n\}$ are monotonic, the one increasing and the other decreasing

(ii) $\{a_n\}$ and $\{b_n\}$ both converge to the same limit.

Sol Suppose $a_1 > b_1$.

$\therefore$ For any two +ve nos $G.M > H.M.$

$\therefore a_n > b_n \quad \forall n \qquad \longrightarrow ①$

Also $a_{n+1} = \sqrt{a_n\cdot b_n} < \sqrt{a_n\cdot a_n} = a_n.$

$\Rightarrow \{a_n\}$ is $\downarrow.$ ie

$a_1 > a_2 > a_3 > a_4 - - - > a_n >$

$\Rightarrow a_1 >> a_n \quad \forall n. \qquad \longrightarrow ②$

Again $b_{n+1} = \dfrac{2a_n b_n}{a_n + b_n} > \dfrac{2b_n\cdot b_n}{b_n + b_n} = b_n \quad [a_n > b_n]$

$\Rightarrow b_{n+1} > b_n$

$\{b_n\}$ is monotonic increasing $b_1 < b_2 < b_3 .. < \underline{\qquad}$

$\longrightarrow ④$

$b_n > b_1 \quad \forall n$

is greater than $\ldots$

From ①, ②, ③ & $\frac{13B}{④}$ we get

$a_1 \geqslant a_n > b_n \geqslant b_1$ ⟶ ⑤

Now since $a_n > b_1 \quad \forall n$

$\Rightarrow \{a_n\}$ is bounded below and being $\forall n$ converges to a limit.

Again ⑤ $\Rightarrow b_n < a_1 \quad \forall n$

$\Rightarrow \{b_n\}$ is bounded above and being $\forall n$ is convergent.

Let $\lim\limits_{n \to \infty} a_n = a$ & $\lim\limits_{n \to \infty} b_n = b$

Now $a_{n+1} = \sqrt{a_n \cdot b_n}$

or $a_{n+1}^2 = a_n b_n$

$\therefore$ we have $\lim\limits_{n \to \infty} a_{n+1} = \lim\limits_{n \to \infty} a_n \cdot \lim\limits_{n \to \infty} b_n$

$a^2 = a.b$

$\Rightarrow a = b$ which proves the result.

Q.28 Two sequences $\{x_n\}$ and $\{y_n\}$ are defined.

inductively by $x_1 = \frac{1}{2}$ & $y_1 = 1$.

and $x_n = \sqrt{x_{n-1} y_{n-1}} \qquad n = 2, 3, \ldots$

$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-2}}\right) \qquad n = 2, 3, 4, \ldots$

prove that $x_{n-1} < x_n < y_n < y_{n-1} \quad n = 2, 3, \ldots$ and deduce that both sequences converge to the same limit ℓ where $\frac{1}{2} < \ell < 1$.

Sol If $0 < a < b$, then $G.M = \sqrt{ab}$

and $H = \left[\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\right]^{-1} = \frac{2ab}{a+b}$

Also $a < H < G < b$

we are given that $\underline{(3)}$

$\frac{1}{2} = x_1 < y_1 = 1.$

$\Rightarrow$ Let $x_{n-1} \geq y_{n-1}$

$x_{n-1} < x_n < y_{n-1}$

$\qquad\qquad\qquad \left[\because x_n = \sqrt{x_{n-1}\, y_{n-1}} \right]$

Further

$x_n < y_n < y_{n-1}$

because y_n is H.M of $x_{n-1} \not< y_{n-1}$.

$\Rightarrow x_{n-1} < x_n < y_n < y_{n-1} \qquad n = 2, 3, \cdots$

$\Rightarrow \{x_n\} \uparrow$ and is bounded above by $y_1 = 1$

The sequence $\{y_n\}$ decreases and is bounded below by $x_1 = \frac{1}{2}$. Hence both the sequence converge.

Suppose $x_n \to l$ as $n \to \infty$ and $y_n \to m$ as $n \to \infty$

Then $l^2 = lm \Rightarrow \boxed{l = m.}$

$\frac{1}{m} = \frac{1}{2}\left(\frac{l+m}{lm} \right) \Rightarrow \boxed{l = m}$

$\underline{Q.29}$ If a sequence $\{a_n\}$ is defined by

$a_{n+1} = 1 + \frac{1}{a_n},\ a_1 > 0,\ a_1 = 1$

$\quad a_{n \geq 2} \qquad a_2 = 2$

prove that the sequence $\{a_n\}$ is cgt and

$\lim_{n \to \infty} a_n = \frac{1 + \sqrt{5}}{2}$

$\underline{Sol}$ $a_3 \leq 1 + \frac{1}{2} < 2.$ $\qquad\qquad a_3 = \frac{3}{2}$

$a_4 = 1 + \frac{1}{a_3} = 1 + \frac{2}{3} < 2.$

Let $a_{n+1} < 2$ for $n \geq 2$

$\qquad\qquad$ is greater

132

Let $a_{k+1} < 2$ for $k > 2$.

$\Rightarrow \dfrac{1}{a_{k+1}} < \dfrac{1}{2}$

$\Rightarrow 1 + \dfrac{1}{a_{k+1}} < 1 + \dfrac{1}{2} < 2$.

$\Rightarrow a_{k+1+1} < 2$

Thus $a_{n+1} < 2$ $\forall n > 2$

Thus $1 < a_n \le 2$ $\forall n$.

$a_n = 1 + \dfrac{1}{a_{n-1}} = 1 + \dfrac{1}{1 + \dfrac{1}{a_{n-2}}}$

$\quad = 1 + \dfrac{a_{n-2}}{1 + a_{n-2}}$

$a_{n+2} - a_n = \left(1 + \dfrac{a_n}{1+a_n}\right) - \left(1 + \dfrac{a_{n-2}}{1+a_{n-2}}\right)$

$\quad = \dfrac{a_n - a_{n-2}}{(1+a_n)(1+a_{n-2})}$

$\dfrac{1 + \frac{a_{n-2}}{1+a_{n-2}}}{a_{n-2}\, a_{n-1}\, a_n\, a_{n+1}\, a_{n+2}}$

$\Rightarrow a_{n+2} - a_n$ have same sign as $a_n - a_{n-2}$.

Now

$a_3 - a_1 = \dfrac{3}{2} - 1 = \dfrac{1}{2} > 0$

so $a_{2k+1} - a_{2k-1} > 0$ $\forall k \in N$

It follows that $\{a_{2k-1}\}$ is monotone increasing sequence.

Similarly $a_4 - a_2 = \dfrac{5}{3} - 2 = \dfrac{-1}{3} < 0$

and so $a_{2k+2} - a_{2k} < 0$ $\forall k \in N$

$\Rightarrow \{a_{2n}\}$ is monotone decreasing sequence

Also $1 \leq a_{2n-1} \leq 2 \qquad \forall\, n \in N$

$\qquad 1 \leq a_{2n} \leq 2 \qquad \forall\, n \in N$

$\Rightarrow \{a_{2n}\}$ and $\{a_{2n-1}\}$ are bounded monotone

sequence and hence cgt.

Let $\quad \ell_1 = \lim\limits_{n \to \infty} a_{2n-1}$

$\qquad \ell_2 = \lim\limits_{n \to \infty} a_{2n}.$

Now $\quad a_n = 1 + \dfrac{a_{n-2}}{1+a_{n-2}} \qquad$ for $n \geq 3$

$a_{2n-1} = 1 + \dfrac{a_{2n-3}}{1+a_{2n-3}}$

$a_{2n} = 1 + \dfrac{a_{2n-2}}{1+a_{2n-2}}$

$\lim\limits_{n \to \infty} a_{2n-1} = 1 + \dfrac{\lim\limits_{n \to \infty} a_{2n-3}}{1+\lim\limits_{n \to \infty} a_{2n-3}}$

$\ell_1 = 1 + \dfrac{\ell_1}{1+\ell_1}$

$\lim\limits_{n \to \infty} a_{2n} = 1 + \dfrac{\lim\limits_{n \to \infty} a_{2n-2}}{1+\lim\limits_{n \to \infty} a_{2n-2}}$

$\ell_2 = 1 + \dfrac{\ell_2}{1+\ell_2}$

Thus ℓ_1 & ℓ_2 both satisfy equation $\ell^2 - \ell - 1 =$

But $1 \leq a_n \leq 2. \quad \forall n.$

But $l = \dfrac{1 \pm \sqrt{5}}{2} \quad \forall n.$

So $l_1, l_2 > 0$

$\Rightarrow l_1 = \dfrac{1+\sqrt{5}}{2} = l_2.$

Q #30 (Gashil) If a sequence $\{a_n\}$ is defined.

$a_2 > a_1 > 0, \quad a_{n+1} = \dfrac{a_n + a_{n-1}}{2} \qquad \forall n \geq 2$

or $a_n = \dfrac{a_{n-1} + a_{n-2}}{2} \quad \forall n \geq 2.$

prove that the sequence converges also find.

Limit

Sol # $a_{n+2} - a_n = \dfrac{1}{2}[a_{n+1} + a_n] - a_n$

$= \dfrac{1}{2}[a_n - a_{n-2}] \longrightarrow ①$

$\therefore \quad a_1 < a_2$

putting $n = 3, 4, 5 \cdots$ in relation.

$a_n = \dfrac{1}{2}[a_{n-1} + a_{n-2}]$

$a_3 = \dfrac{1}{2}[a_2 + a_1]$

$a_4 = \dfrac{1}{2}[a_3 + a_2]$

$a_5 = \dfrac{1}{2}[a_4 + a_2]$

$$a_m = \frac{1}{2}[a_{m-1} + a_{m-2}]$$

$$\therefore \ a_1 < a_2$$

$$a_3 = \frac{1}{2}[a_1 + a_2]$$

But $a_3 = \frac{1}{2}[a_1 + a_2] < \frac{1}{2}[a_1 + a_2] = a_2$

$$\Rightarrow a_1 < a_3 < a_2$$

$$a_3 < a_4 < a_2$$

$$a_3 < a_5 < a_4$$

$$a_5 < a_6 < a_4$$

Thus it appears.

$$a_1 < a_3 < a_5 - - - -$$

$$a_2 > a_4 > a_6 - - - -$$

from ① let $n = 2m$

$$a_{2m+2} - a_{2m} = \frac{1}{2}\{a_{2m+1} - a_{2m}\}$$

$$\therefore a_{2m+2} < a_{2m} \Rightarrow a_{2m+1} - a_{2m} \gtrless 0$$

$$\Rightarrow a_{2m+1} - a_{2m} < 0$$

$$\Rightarrow a_{2m+1} < a_{2m}$$

but

$$\Rightarrow a_{2m} < a_{2m-2} < \ldots < \ldots \ a_4 < a_2.$$

∴ every odd term is less than every even term

i.e $\{a_{2m+1}\}$ is increasing and bounded above

by a_2 and is therefore convergent

Similarly even term subsequence $\{a_{2n}\}$ is convergent. We show that both converge to same limit

Let $\lim_{n \to \infty} a_{2n+1} = l_1$ & $\lim_{n \to \infty} a_{2n} = l_2$

From recursion relation.

$$a_{2n} = \frac{1}{2}[a_{2n-1} + a_{2n-2}]$$

$$\lim_{n \to \infty} a_{2n} = \frac{1}{2}[\lim_{n \to \infty} a_{2n-1} + \lim a_{2n-2}]$$

$$l_2 = \frac{1}{2}[l_1 + l_2]$$

$$2l_2 = l_1 + l_2$$

$$l_1 = l_2$$

Thus $\{a_n\}$ is cgt.

Now from $a_n = \frac{1}{2}[a_{n-1} + a_{n-2}]$

$$a_{n+1} = \frac{1}{2}[a_{n} + a_{n-1}]$$

$$a_3 = \frac{1}{2}[a_2 + a_1]$$

$$a_4 = \frac{1}{2}[a_3 + a_2]$$

$$a_5 = \frac{1}{2}[a_4 + a_3]$$

$$--$$

$$a_k = \frac{1}{2}[a_{k-1} + a_{k-2}]$$

$$a_{k+1} = \frac{1}{2}[a_k + a_{k-1}]$$

Adding all these

$$a_3 + a_4 + a_5 + \ldots + a_k + a_{k+1} = \frac{1}{2}[a_2 + a_1 + a_3 + a_2$$
$$+ a_4 + a_3 + \ldots$$
$$\ldots \quad a_{k-1} + a_{k-2}$$
$$+ a_k + a_{k-1}]$$

$$= \frac{1}{2}[a_1 + 2a_2 + 2a_3 + 2a_4$$
$$+ \ldots - 2a_{k-1} + a_k]$$

$$\frac{1}{2}a_k + a_{k+1} = \frac{1}{2}[a_1 + 2a_2]$$

$\Rightarrow$ Taking limit when $k \to \infty$

$$\frac{1}{2}l + l = \frac{1}{2}[a_1 + 2a_2]$$
$$\frac{3}{2}l = \frac{1}{2}[a_1 + 2a_2]$$
$$l = \frac{1}{3}[a_1 + 2a_2]$$

$$= a \left[\frac{1+a_{n-3} - 1 - a_{n-1}}{(1+a_{n-1})(1+a_{n-3})} \right]$$

$$= -a \left(\frac{a_{n-1} - a_{n-3}}{(1+a_{n-1})(1+a_{n-3})} \right) \rightarrow ②$$

$$= -a \left[\frac{a}{1+a_{n-2}} - \frac{a}{1+a_{n-4}} \right]$$

$$= -a \left[\frac{(1+a_{n-1})(1+a_{n-3})}{} \right]$$

$$= \frac{a^2 (a_{n-2} - a_{n-4})}{(1+a_{n-1})(1+a_{n-3})(1+a_{n-2})(1+a_{n-4})} \rightarrow ③$$

③ $\Rightarrow$ that $a_n - a_{n-2}$ & $a_{n-2} - a_{n-4}$ have same
signs. So even and odd terms form seperate
monotone sequences

For ② we note that if odd numberd sub-
sequence form a monotone decreasing sub-sequence
. Then subsequence of even no terms form a sub-sequence
of increasing terms and vice versa.

139

Since. Every term of the sequence is +ve.

$$a - a_n = a - \frac{a}{1+a_{n-1}} = \frac{a\, a_{n-1}}{1+a_{n-1}} > 0$$

$\Rightarrow \quad a - a_n > 0 \quad$ ∀n.

$\Rightarrow \quad a_n < a \quad$ ∀n

Thus $\quad 0 < a_n < a$.

OR

$$a_n = \frac{a}{1+a_{n-1}} < a \qquad \because \ a_{n-1} > 0$$

Thus monotone increasing sub-sequence is bounded. above by a and the monotone decreasing subsequence is bounded below by 0

Hence the two sub-sequence converge.

Let even no terms converges to l_1 & odd no term converge to l_2.

Then $\displaystyle \lim_{n\to\infty} a_n = \frac{a}{1+\lim_{n\to\infty} a_{n-1}}$

we get

(a) For n even $\quad l_1 = \dfrac{a}{1+l_2} \quad$ or $\quad l_1 l_2 + l_1 = a$.

(b) For n odd $\quad l_2 = \dfrac{a}{1+l_1} \quad$ or $\quad l_1 l_2 + l_2 = a$.

$\Rightarrow \quad l_1 l_2 + l_1 = l_1 l_2 + l_2$

$\Rightarrow \quad \boxed{l_1 = l_2}$

Q#32
140

A sequence $\{a_n\}$ is defined as $a_1 = 1$

$a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$ $n \geq 1$ Show that $\{a_n\}$

Converges and find limit

Sol

$a_1 = 1$ $a_2 = \dfrac{4+3a_1}{3+2a_1} = 7/5 > 1$

$a_1 > a_2$

Let $a_{n+1} > a_n$.

Then $a_{n+2} - a_{n+1} = \dfrac{4+3a_{n+1}}{3+2a_{n+1}} - \dfrac{4+3a_n}{3+2a_n}$

$= \dfrac{a_{n+1} - a_n}{(3+2a_{n+1})(3+2a_n)} > 0$ $\left[\because a_{n+1} > a_n \atop a_n > 0 \; \forall n \right]$

$a_{n+2} - a_{n+1} > 0$

$\Rightarrow a_{n+2} > a_{n+1}$

By mathematical induction $\{a_n\}$ is increasing

$a_{n+1} = \dfrac{4+3a_n}{3+2a_n} = \dfrac{3}{2} - \dfrac{1}{2(3+2a_n)}$

$= \dfrac{3}{2} - (\text{a +ve quantity less than 1})$ $[a_n \geq 1 \; \forall 1]$

$< \dfrac{3}{2}$ $\Rightarrow a_{n+1} < \dfrac{3}{2}$ $\forall n$.

$\Rightarrow \{a_n\}$ is bounded above.

146

Thus Seq is bounded monotone and hence convergent. Let $\lim_{n\to\infty} a_n = l$.

Then $a_{n+1} = \dfrac{4+3a_n}{3+2a_n}$

$\Rightarrow l = \dfrac{4+3l}{3+2l}$

$3l + 2l^2 = 4 + 3l$

$2l^2 = 4$ $\Rightarrow l^2 = 2$ $\Rightarrow l = \dfrac{4}{2} = 2$

$\Rightarrow l = \pm\sqrt{2}$. But l cannot be -ve. $l = \sqrt{2}$

Q.33 A sequence is defined by

$a_{n+1} = 1 + \frac{1}{2} a_n$

$a_0 > 1$ $\qquad a_1 = \frac{3}{2}$

Show that the sequence converges by showing the sequence is bounded and find its limit.

Sol.

$a_1 = \frac{3}{2} = 1.5$

1.75

$a_0 = 1$

Limit

$a_2 = ?$ $= 1 + \frac{1}{2} a_1 = 1 + \frac{1}{2} \cdot \frac{7}{4} = \frac{15}{8} = 1.88$

$a_3 = 1 + \frac{1}{2} a_2$

appears to increase.

The sequence
Hence it is possible only if

$a_{n+1} > a_n$

$1 + \frac{1}{2} a_n > a_n$

$\Rightarrow 1 > \frac{1}{2} a_n \Rightarrow 2 > a_n$

$a_n < 2 \quad \forall n$

147

⟹ Sequence is bounded above by 2.
Hence the sequence is bounded, and thus convergent

Let $\lim_{n\to\infty} a_n = l$.

⟹ $l = 1 + \frac{1}{2} l$ ⟹ $l = 2$.

Thus $\lim_{n\to\infty} a_n = 2$

Theorem: If $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$

Proof: **Case I** Let $0 < a < 1$

Then $\{a^n\}$ is a bounded monotone sequence and thus convergent.

Let $\lim_{n\to\infty} a^n = l$.

Then $l = \lim_{n\to\infty} a^{n+1} = \lim_{n\to\infty} a \cdot a^n$

$l = a l$

⟹ $l(1-a) = 0$ ⟹ $l = 0$ or $a = 1$

If $l \neq 0$, then $a = 1$ which is impossible.

because $|a| < 1$ ⟹ $-1 < a < 1$

Therefore $\lim_{n\to\infty} a^n = 0$ if $0 < a < 1$

Case II Let $-1 < a < 0$

Then $0 < -a < 1$

Now $(-a)^n = (-1)^n a^n$.

$\{(-1)^n\}$ is bounded & $\lim_{n\to\infty} a^n = 0$

$\Rightarrow \lim_{n\to\infty} (-1)^n a^n = 0$

$\Rightarrow \lim_{n\to\infty} (-a)^n = 0$

$\lim_{n\to\infty} a^n = \lim_{n\to\infty} [(-1)(-a)]^n$

$= \lim_{n\to\infty} (-1)^n (-a)^n = 0$

$\Rightarrow \lim_{n\to\infty} a^n = 0$

__Case II__ If $a > 0$, then $\lim_{n\to\infty} a^n = 0$

__OR__

If $a > 0$, then $\lim_{n\to\infty} a^n = 0$

Let us as assume that $0 < |a| < 1$

Let $\epsilon > 0$

$|a^n - 0| < \epsilon$

$|a^n| < \epsilon \Rightarrow |a|^n < \epsilon$

$n \ln|a| < \ln \epsilon$

$n > \dfrac{\ln \epsilon}{\ln|a|}$

$\therefore \ln|a| < 0$

because $0 < |a| < 1$

This in equally provides us a clue
to choose n^{th} (fixed). Let us consider

__Case I__ If $\epsilon < 1$

149

Then $\ln \varepsilon < 0$ and let $n_1 > \dfrac{\ln \varepsilon}{\ln|a|}$

Such that
$$|a^n - 0| < \varepsilon \qquad \forall\, n > n_1.$$
$$\Rightarrow \lim_{n \to \infty} a^n = 0$$

Case II: If $\varepsilon > 1$, then $\lim \varepsilon > 0$
and $\dfrac{\ln \varepsilon}{\ln|a|} < 0$

In this case we can choose an $n_1 + ve$
and will have $\qquad \forall\, n > n_1)$
$$|a^n - 0| < \varepsilon \qquad \forall\, n > n_1)$$
$$\Rightarrow \lim_{n \to \infty} a^n = 0$$

Theorem: If $a > 0$, then $\displaystyle\lim_{n \to \infty} a^{1/n} = 1$

Prob # Case-I: Let $a > 1$
$a^{1/n} > 1$
Then $a^{1/n} > 1$
$\{a^{1/n}\}$ is bounded below by 1.
Also $a > a^{1/2} > a^{1/3} > a^{1/4} > ---$
$\Rightarrow \{a^{1/n}\}$ is bounded and decreasing
$\Rightarrow \{a^{1/n}\}$ is convergent
and hence $\displaystyle\lim_{n \to \infty} a^{1/n} = l.$
$\therefore$ Let $\displaystyle\lim_{n \to \infty} a^{1/n} = l.$
$a^{1/n} > 1$
$\Rightarrow \displaystyle\lim_{n \to \infty} a^{1/n} > 1$

$$\Rightarrow \lim_{n \to \infty} a_n \quad \frac{a_{2n+1}}{a_{2n}} = l \cdot l \to \lim_{n \to \infty} (a^{1/n})^2 \to l^2$$

The Sub-sequence $\{a_{2n}\}$ & $\{a_{2n+1}\}$ converge to same limit l^2 hence.

$$l = l^2$$
$$l(l-1) = 0$$
$$l = 0 \quad \text{or} \quad l = 1$$

Since $l \neq 0$, $l = 1$

$$\Rightarrow \lim_{n \to \infty} a^{1/n} = 1$$

OR

Let $a > 1$, then $a^{1/n} > 1 \quad \forall n$

Let $a^{1/n} = 1 + h_n$ when $h_n > 0$

$$a = (1 + h_n)^n = 1 + n h_n + \frac{n(n-1)}{2!} h_n^2 \ldots h_n^2 \cdot h_n$$

$$\geqslant 1 + n h_n$$

$$\Rightarrow \frac{a-1}{n} \geqslant h_n$$

$$\Rightarrow 0 < h_n < \frac{a-1}{n} \quad \forall n.$$

$$\Rightarrow \lim_{n \to \infty} 0 = \lim_{n \to \infty} \frac{a-1}{n} = 0$$

By Squeeze principle

$$\lim_{n \to \infty} h_n = 0$$

$$\lim_{n \to \infty} a^{1/n} = \lim_{n \to \infty} (1 + h_n) = 1 + 0 = 1$$

If $a = 1$, $1^{1/n} = 1$

, then $\lim\limits_{n \to \infty} a^{1/n} = 1$

Case II: If $0 < a < 1$

Then $\dfrac{1}{a} > 1$

$\lim\limits_{n \to \infty} \left(\dfrac{1}{a}\right)^{1/n} = \lim\limits_{n \to \infty} \dfrac{1}{a^{1/n}} \geq 1$

$\Rightarrow \lim\limits_{n \to \infty} a^{1/n} \geq 1$

Theorem:# Discuss the nature of the sequence $\{a^n\}$ for $a \in R$.

So: The behaviour of the sequence $\{a^n\}$ depends upon the value of a

Case I: Let $a > 1$

Then $a = 1+h$

$a^n = (1+h)^n = 1+nh + \dfrac{n(n-1)}{2}h^2 + \dots + h^n$ where $h > 0$

$a^n \geq 1+nh$

as $n \to \infty$ $1+nh \to \infty$

$\Rightarrow a^n \to \infty$ as $n \to \infty$

$\Rightarrow \{a^n\}$ diverges

Case II: For $a = 1$

$a^n = 1$

$\Rightarrow \{a^n\}$ is a constant sequence and converges

Case III Let $0 < a < 1$

Then $\frac{1}{a} > 1$

$\frac{1}{a} = 1+h$ for some $h > 0$

$\frac{1}{a^n} = (1+h)^n > nh$

$\Rightarrow 0 < a^n < \frac{1}{nh}$

By squeeze play

$\lim\limits_{n\to\infty} a^n = 0$

Case II If $a > 0$

$\lim\limits_{n\to\infty} a^n = 0 \Rightarrow \{a^n\}$ converges to 0

Case I Let $-1 < a < 0$

put $a = -b$, then

$-1 < a < 0 \Rightarrow -1 < -b < 0$

$\Rightarrow 0 < b < 1$

$b^n \to 0$ as $n \to \infty$

$\lim\limits_{n\to\infty} a^n = \lim\limits_{n\to\infty} (-b)^n = \lim\limits_{n\to\infty} (-1)^n a^n = 0$

$\Rightarrow \{a^n\}$ converges to 0

Case III Let $a = -1$

$a^n = (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$

The sequence $-1, 1, -1, 1,$ oscillates finitely

$\Rightarrow \{a^n\}$ oscillates finitely

Case VIII

$a = -b$ Let $a = -b$

$a^n = (-1)^n$

$a < -1 \Rightarrow -a < -1 \Rightarrow (-a)^n \to \infty$

By $b^n \to \infty$ as $n \to \infty$

$a^n = (-b)^n = \begin{cases} -b^n & \text{if } n \text{ is odd} \\ b^n & \text{if } n \text{ is even} \end{cases}$

$a^n \to \infty$ as $n \to \infty$ when n is odd

$a^n \to \infty$ as $n \to \infty$ when n is even

$\Rightarrow \{a^n\}$ oscillates infinitely

Hence $\{a^n\}$ converges when $-1 < a < 1$

Lemma

Let a & b be numbers such that $0 \le a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b-a} < (n+1) b^n \quad \text{i.e.}$$

or

$$b^n \times \{ b - (n+1)(b-a)\} < a^{n+1}$$

Proof

By actual division. $0 \le a < b$

$$\frac{b^{n+1} - a^{n+1}}{b-a} = a^0 b^n + a b^{n-1} + a^2 b^{n-2} + \cdots + a^{n-2} b^2 + a^{n-1} b + a^n$$

(n terms)

$$< b^n + a b^{n-1} + a^2 b^{n-2} + \cdots + a^{n-3} b^2 + a^{n-1} b + b^n \quad (\because a < b)$$

$$= (n+1) b^n$$

$$\Rightarrow \frac{b^{n+1} - a^{n+1}}{b-a} < (n+1) b^n$$

$$\Rightarrow b^{n+1} - a^{n+1} < (n+1) b^n [b-a]$$

$$b^{n+1} - (n+1)b^n(b-a) < a^{n+1}$$

$$b^n[b - (b-a)(n+1)] < a^{n+1}$$

Theorem

prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is bounded and increasing

Q^{ℓ}

Let $e_n = \left(1+\frac{1}{n}\right)^n$, prove that $\{e_n\}$ is increasing and bounded.

Proof

We know that for $0 \leq a < b$

$$b^n[b - (n+1)(b-a)] < a^{n+1}$$

Taking $a = 1 + \frac{1}{n+1}$ & $b = 1 + \frac{1}{n}$

$b > a$ $\therefore \frac{1}{n+1} < \frac{1}{n}$

$$\left(1+\frac{1}{n}\right)^n\left[\left(1+\frac{1}{n}\right) - (n+1)\left(1+\frac{1}{n} - 1 - \frac{1}{n+1}\right)\right] < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n}\right)^n\left[\left(1+\frac{1}{n}\right) - (n+1)\left(\frac{1}{n} - \frac{1}{n+1}\right)\right] < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n}\right)^n\left[\left(1+\frac{1}{n}\right) - (n+1)\left(\frac{n+1-n}{n(n+1)}\right)\right] < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n}\right)^n\left[1+\frac{1}{n} - \frac{1}{n}\right] < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\Rightarrow e_n < e_{n+1} \quad \forall n.$$

Q.9

$\Rightarrow \{e_n\}$ is $\uparrow$

Let $a = 1$ $\qquad b = 1 + \dfrac{1}{2n}$

$\left(1 + \dfrac{1}{2n}\right)^n \left\{1 + \dfrac{1}{2n} - (n+1)\left(1 + \dfrac{1}{2n}7\right)\right\} < 1^{n+1}.$

$\left(1 + \dfrac{1}{2n}\right)^n \left[1 + \dfrac{n}{2n} - \dfrac{1}{2} - \dfrac{1}{2n}\right] < 1$

$\left(1 + \dfrac{1}{2n}\right)^n \left[\dfrac{n}{2n}\right] < 1$

$\Rightarrow \left(1 + \dfrac{1}{2n}\right)^n < 2.$

$\Rightarrow \left(1 + \dfrac{1}{2n}\right)^{2n} < 4. \Rightarrow e_{2n} < 4$

Since $\{e_n\}$ is increasing and $2n > n$.

$e_n < e_{2n} < 4 \quad \forall n.$

$\Rightarrow \{e_n\}$ is bounded.

Thus $\{e_n\}$ is convergent.

If $n = 1$ $\qquad e_1 = (1+1)^1 = 2.$

$\Rightarrow 2 \leq e_n < 4 \quad \forall n.$

OR

$\therefore$ n is true integer, by binomial theorem.

$e_n = \left(1 + \dfrac{1}{n}\right)^n = 1 + n \cdot \dfrac{1}{n} + \dfrac{n(n-1)}{2!} \cdot \dfrac{1}{n^2} + \cdots + \dfrac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \dfrac{1}{n^n}.$

$$e_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$
$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

changing n to $n+1$

$$e_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots$$
$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$
$$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

note that e_n has $n+1$ terms & e_{n+1} has $n+2$ terms and one more term.

Also $\dfrac{1}{n+1} < \dfrac{1}{n} \Rightarrow -\dfrac{1}{n+1} > -\dfrac{1}{n}$

$\Rightarrow 1 - \dfrac{1}{n+1} > 1 - \dfrac{1}{n}$.

Similarly $1 - \dfrac{k}{n+1} > 1 - \dfrac{k}{n}$ for $k=1,2,\ldots n-1$

$\Rightarrow$ each term in e_n after the 1st two terms is less than the corresponding term in e_{n+1} and e_{n+1} has also one additional term. It comes out

$$e_{n+1} \gg e_n$$

$\Rightarrow \{e_n\}$ is $\uparrow$

To show that $\{e_n\}$ is bounded above we have

$$= 1 + n\cdot\frac{1}{n} + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots$$
$$\cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$< 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} + \cdots + \frac{1}{n!}$$
$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}} + \cdots + \frac{1}{2^{n-1}}$$

$$\therefore 1 - p_n < 1$$

$$\therefore \frac{1}{p_n} \leq \frac{1}{2^{p-1}}$$

(157)

$$= 1 + \frac{1-(1/2)^n}{1-1/2} = 1 + 2[1-(1/2)^n] < 3$$

$\Rightarrow e_n < 3 \quad \forall n.$

for $n=1, \quad e_1 = 2.$

for $n>1$

$\qquad 2 < e_n < 3 \quad \forall n > 1$

Thus $\{e_n\}$ bounded monotone and hence convergent

Note By refining our estimates we can find closer rational approximations to e but we can not evaluate e. Since e is an irrational number. However it is possible to evaluate e to as many decimal places as desired.

Theorem # Prove that e is irrational

Proof # Let e be a rational number

$$e = \frac{p}{q} \qquad \text{where } p, q \in N$$
$$\text{and } q > 1$$

Let $S_q = \sum_{k=1}^{q} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}$

Since $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

So $0 < e - S_q = \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \frac{1}{(q+3)!} + \cdots$

$$= \frac{1}{(q+1)!} \left[1 + \frac{1}{q+2} + \frac{1}{(q+2)(q+3)} + \cdots \right]$$

$$< \frac{1}{(q+1)!} \left[1 + \frac{1}{q+1} + \frac{1}{(q+1)^2} + \cdots \right]$$

$$= 1 + \frac{1-(1/2)^n}{1-1/2} = 1 + 2[1-(1/2)^n] < 3$$

$\Rightarrow e_n < 3$ ∀n

for n=1, $e_1 = 2$.

for n>1

$2 < e_n < 3$ ∀n>1

Thus $\{e_n\}$ bounded monotone and hence convergent

Note: By refining our estimates we can find closer rational approximations to e but we can not evaluate e since e is an irrational number. However it is possible to evaluate e to as many decimal places as desired.

Theorem# Prove that e is irrational

Proof# Let e be a rational number

$$e = \frac{p}{q} \quad \text{where } p, q \in N$$

and $q > 1$

Let $S_q = \sum_{k=1}^{q} \frac{1}{k!}$

Since $e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$

$S_q = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{q!}$

So $0 < e - S_q = \frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\frac{1}{(q+3)!}+\cdots$

$= \frac{1}{(q+1)!}\left[1+\frac{1}{q+2}+\frac{1}{(q+2)(q+3)}+\cdots\right]$

$< \frac{1}{(q+1)!}\left[1+\frac{1}{q+1}+\frac{1}{(q+1)^2}+\cdots\right]$

$\Rightarrow a_n < a_m < -k \quad \forall n \geq m$

$\Rightarrow a_n < -k \quad \forall n \geq m$

$\Rightarrow \lim_{n\to\infty} a_n = -\infty \Rightarrow \{a_n\}$ diverges to $-\infty$

$$= \frac{1}{(q+1)!}\left[\frac{1}{1-\frac{1}{q+1}}\right] \qquad S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{(q+1)!} \times \frac{q+1}{q} = \frac{1}{(q!)q}$$

$$\Rightarrow 0 < (e - S_q) < \frac{1}{q!\,q}$$

$$\Rightarrow 0 < (e - S_q)q! < \frac{1}{q} < 1 \qquad (\because q! \geq 1)$$

$$0 < e.q! - S_q(q!) < 1.$$

$$\therefore q.e = p \qquad \text{by } p/q = e$$

$$\Rightarrow e.q! = e.q(q-1)! = p(q-1)! \text{ is an integer}$$

Also

$$(q!).S_q = q!\left[1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right]$$

$$= q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + 1 \text{ is}$$

an integer for each $q > 1$

$$\Rightarrow (e - S_q)q! \text{ is an integer lying b/w}$$

0 and 1 which is a contradiction. because there

is no integer bet. 0 & 1.

Hence $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

is an ~~integer~~ irrational no.

$$2^{k-1} e \cdots + \frac{1}{2^{n-1}} \qquad \therefore \frac{1}{P_i} < \frac{1}{2^{p-1}}$$

(-1)

Theorem #(i) Every monotonically increasing sequence which is not bounded above diverges to $+\infty$ ie diverges properly.

(ii) Every monotonically decreasing sequence which is not bounded below diverges to $-\infty$

Proof #(i) Let $\{a_n\}$ is $\uparrow$ which is no bounded above. Then given any $k>0$, however large, $\exists$ a +ve integer m such that

$$a_m > k.$$

$\therefore$ $\{a_n\}$ is $\uparrow$

$$a_n \geq a_m > k \qquad \forall n \geq m.$$

$$\Rightarrow a_n > k. \qquad \forall n \geq m.$$

Thus for every real no $k>0$, however large, we have a +ve integer m s.t that

$$a_n > k. \qquad \forall n \geq m.$$

$$\Rightarrow \lim_{n \to \infty} a_n = \infty$$

$\Rightarrow \{a_n\}$ diverges to $+\infty$

(ii) Let $\{a_n\}$ which is not bounded below. Then given any $k>0$, large, $\exists$ a +ve integer m s.t

$$a_m < -k$$

$\therefore$ $\{a_n\}$ $\downarrow$

$$a_n \leq a_m < -k. \qquad \forall n \geq m.$$

$$\Rightarrow a_n < -k \qquad \forall n \geq m.$$

$$\Rightarrow \lim_{n \to \infty} a_n = -\infty \Rightarrow \{a_n\} \text{ diverges to } -\infty$$

Theorem

Every monotone sequence either converges or diverges OR A monotone sequence is never oscillatory.

Proof

Let $\{a_n\}$ be a monotone sequence, then either $\{a_n\}\uparrow$ or $\{a_n\}\downarrow$

Case I: Let $\{a_n\}\uparrow$

If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to l.u.b. If $\{a_n\}$ is not bounded, then it diverges to $+\infty$

Case II: Let $\{a_n\}\downarrow$

If $\{a_n\}$ is bounded below, then $\{a_n\}$ is cgt to g.l.b. If $\{a_n\}$ is not bounded below, then $\{a_n\}$ is dgt to $-\infty$

Nested Intervals

A sequence $\{I_n = [a_n, b_n]\}$ of closed intervals is called nested if

$$I_{n+1} \subseteq I_n \quad ie$$

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots I_n \supseteq I_{n+1} \cdots$$
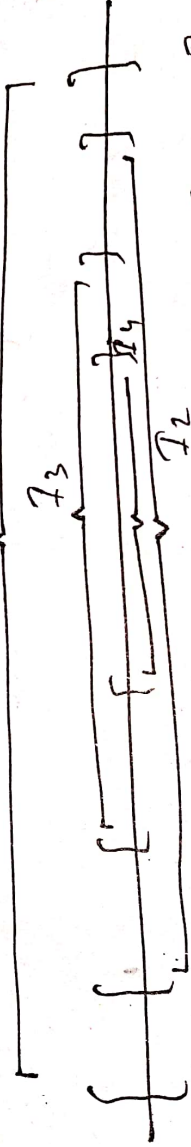
e.g $I_n = [0, 1/n] \quad \forall n \in N$, then $I_n \supseteq I_{n+1}$

In this case 0 belong to all I_n and 0 belong to $\cap I_n = \{0\}$

is the only such common point ie $\cap I_n = \{0\}$ $n \in N$

155

In general, a nested interval sequence need not have common point e.g

if $I_n = (0, 1/n)$ $n \in N$, this sequence $\{I_n\}$ is nested but there is no common point.

because for every $x > 0$ $\exists$ $m \in N$

such that

$$1/m < x \quad \text{(Archimedean)}$$

so that $x \notin I_n$. Similarly the sequence of intervals $K_n = (n, \infty)$ $n \in N$ is nested and have no common point.

However, it is an important property of R that every closed sequence of closed, bounded intervals does have a common point

Nested Interval Property

(Cantor's Nested interval theorem)

Let $\{I_n = [a_n, b_n]\}$ be a sequence of closed intervals such that

(a) $I_{n+1} \subseteq I_n$ ie sequence of nested intervals

(b) $\lim_{n \to \infty} (b_n - a_n) = 0$. Then $\bigcap_{n=1}^{\infty} I_n$ contains unique point i.e c there is exactly one real no common to all intervals I_n.

Proof# Since $I_{n+1} \subseteq I_n$

$$\Rightarrow [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \quad \forall n \in N$$

a₁ a₂ a₃ aₙ aₙ₊₁ ... bₙ₊₄ bₙ b₂ b₁
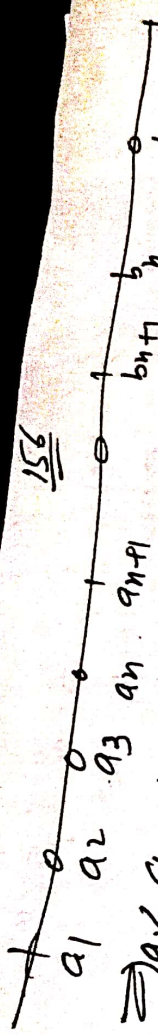
$$a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall n$$

$\Rightarrow \{a_n\}$ is bounded monotonic increasing and $\{b_n\}$ is bounded monotone decreasing sequence.

Thus $\{a_n\}$ & $\{b_n\}$ Converge.

Let $\lim_{n \to \infty} a_n = a$ (lub) $\lim_{n \to \infty} b_n = b$ (glb)

$\therefore b_n = (b_n - a_n) + a_n$

$\Rightarrow \lim_{n \to \infty} b_n = \lim_{n \to \infty} (b_n - a_n) + \lim_{n \to \infty} a_n$

$b = 0 + a \qquad \left[\because \lim_{n \to \infty} (b_n - a_n) = 0 \right]$

$b = a = x \; (say)$

Now x is lub of $\{a_n\}$ &

Also x is glb of $\{b_n\}$

$\Rightarrow a_n \leq x \quad \forall n$ in A

$x \geq a_n \quad \forall n$ in A

$x \in [a_n, b_n] \quad \forall n$ in A

$x \in [a_n, b_n]$ in A

$\Rightarrow x \in \bigcap_{n=1}^{\infty} D_n$

$\Rightarrow x$ is a number common to all intervals.

__Uniqueness__ : If possible let x, y be two distinct numbers common to all the intervals.

$\Rightarrow x \in [a_n, b_n]$ & $y \in [a_n, b_n] \quad \forall n$

Then

of $x > y$, then $a_n \leq y < x \leq b_n$

$$\cdots + \frac{1}{2^{n+1}} x \cdots + \frac{1}{2^{p-1}} \qquad \therefore \frac{1}{p_i} \leq \frac{1}{2^{p-1}}$$

$\Rightarrow$

$\Rightarrow y-x \leq b_n - a_n \quad \forall n$

Let $\varepsilon = y - x > 0$, Then

$$b_n - a_n \geq \varepsilon \quad \forall n. \quad —①$$

Also $\lim\limits_{n \to \infty} (b_n - a_n) \geq 0$

$\Rightarrow$ for $\varepsilon = y - x$ there must exist an integer m such that $|(b_n - a_n) - 0| < \varepsilon \quad \forall n \geq m.$

$\Rightarrow b_n - a_n < \varepsilon \quad \forall n \geq m$

which contradicts ①. Hence x is only element common to all intervals.

Note # The word closed in above theorem cannot be dropped ie. the intersection of a decreasing sequence of open intervals may be empty.

e.g. $I_n = (0, \frac{1}{n}) \quad \forall n \in N$, Then $\bigcap\limits_{n \geq 1}^{\infty} I_n = \phi$

Nested Interval Property

If $I_n = [a_n, b_n]$, $n \in N$ is a nested sequence of closed bounded intervals, then $\exists$ a no $x_0 \in R$ such that $x_0 \in I_n \quad \forall n \in N.$

Proof: Intervals are nested, we have

$$I_n \subseteq I_1, \quad \forall n \in N.$$

So that $a_n \leq b_1 \quad \forall n$ & $a_1 \leq b_n \quad \forall n$

Hence the non-empty set $\{a_n : n \in N\}$ is bounded above and let x_0 be lub of this set

Then $a_n \leq x_0 \quad \forall n$
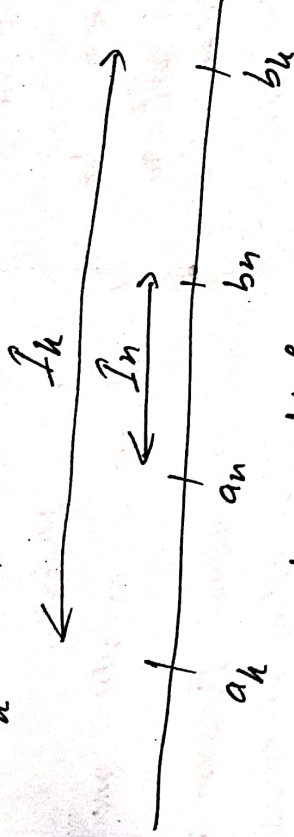
→ finitely many peak points (years) or infinite

we claim that $x_0 \geq b_n$ by assuming that

Because for any particular $x_0 \geq b_n$ is an

upper bound of $\{a_k : k \in N\}$.

If $n \leq k$, then $I_n \supseteq I_k$, we have.

$$a_k \leq b_k \leq b_n$$

If $k < n$, then Since $I_k \supseteq I_n$, we have.

$$a_k \leq a_n \leq b_n$$

$$I_k$$

Thus $a_k \leq b_n \quad \forall k$

$\Rightarrow b_n$ is an upper bound of the set $\{a_k : k \in N\}$

Hence $x_0 \leq b_n$ for each $n \in N$

$\Rightarrow a_n \leq x_0 \leq b_n \quad \forall n$

$\Rightarrow x_0 \in \bigcap_{n=1}^{\infty} I_n$

<u>Note</u>: x_0 above may not be unique.

Peak Point and Peak of a sequence.

A natural no m is called a peak point

of the sequence $\{a_n\}$ if $f_n \not> a_m$

$$a_n \leq a_m \qquad f_n \geq m$$

and the term (a_m) is called a peak of

sequence. i.e a_m is never exceeded by any

term that follows it in the sequence

Note (a) in a decreasing sequence every term is a peak and every natural no is a peak point.

(b) In an increasing sequence no term is peak and no natural no is a peak point

e.g $a_n = \frac{1}{n}$ when $n \leq 5$

Then 1, 2, 3, 4, 5 are five peak points

$a_n = -n$ when $n > 5$

1, 2, 3, 4, 5 are five peak points

(ii) if $a_n =$ when $n = 1, 2, ... m$

$= -1$ when $n > m$

Then m is only peak point.

(iii) if $a_n = \frac{1}{n}$, then every natural no is a peak point

because for any natural no m, there for $n > m$

$\frac{1}{n} < \frac{1}{m}$ i.e $a_n < a_m$ for $n > m$

Thus a sequence may have no peak point, a finite no of peak points or an infinite no of peak points.

Monotone Subsequence Theorem:

Theorem # Every sequence of real nos contain a monotone subsequence.

Proof # Let $\{a_n\}$ be any sequence. sequence $\{a_n\}$ may have no peak point (peak) or infinitely many peak points (peaks) or finitely many peak points.

...eak points.

If The sequence has no peak point and

...me no peak.

...nce 1 is not a peak point, $\exists$ a natural

$n_2 > 1$ such that $a_{n_2} > a_1$

n_2 is not a peak point, $\exists$ a natural no

$n_3 > n_2$ is that $a_{n_3} > a_{n_2}$

Repeating the above argument, we get a

sub-sequence $\{a_{n_k}\}$ such that $n = 1$

$a_{n_1} < a_{n_2} < a_{n_3} < \text{-----}$

is the sequence $\{a_n\}$ contains a monotoni...

increasing subsequence

The sequence $\{a_n\}$ has a finite no of

peak points.

Let m be the largest peak point

Set n_1 be a natural no s. that

a_m is a peak. Set n_1 is not a peak point

m, then n_1 is not a peak point $a_{n_2} > a_{n_1}$

$\exists$ a natural no $n_2 > n_1$ such that

a_{n_2} is not a peak point

$\exists$ a natural no $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$

repeating the above argument we get a

subsequence $\{a_{n_k}\}$ such that

$a_{n_1} < a_{n_2} < a_{n_3} < \text{-----}$ monotone increasing

$\{a_n\}$ contains a monotone increasing

sequence.

The sequence $\{a_n\}$ has an infinite no

...eak points.

...term (a_m) is called a peak $\exists$

i.e. a_m is never exceeded by any

that follows it in the sequence

$\rightarrow n$

Peak Points.

Case I The sequence has no peak point and.

Since no peak.

Hence 1 is not a peak point, ∃ a natural no $n_2 > 1$ such that $a_{n_2} > a_1$

∴ n_2 is not a peak point, ∃ a natural no $n_3 > n_2$ s.that $a_{n_3} > a_{n_2}$

Repeating the above argument, we get a Sub-sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \;--\;--\;-- \qquad n=1$$

Thus the sequence $\{a_{n_k}\}$ contains a monotonically increasing subsequence.

Case II # The sequence $\{a_n\}$ has a finite no of peak points.

Let m be the largest peak point and a_m is a peak. Let n_1 be a natural no s.that $n_1 > m$, then n_1 is not a peak point

∴ ∃ a natural no $n_2 > n_1$ such that $a_{n_2} > a_{n_1}$

Again n_2 is not a peak point

∴ ∃ a natural no $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$

Repeating the above argument we get a Sub-sequence $\{a_{n_k}\}$ such that

$$a_{n_1} < a_{n_2} < a_{n_3} < \;--\;--$$

Thus $\{a_n\}$ contains a monotone increasing Sub-sequence.

Case III: The sequence $\{a_n\}$ has an infinite no of peak points.

and the term (a_m) is called ... never exceeded by any ... sequence. i.e a_m is never ... term that follows it in the sequence

161

Let the peak points be $n_1, n_2, n_3 \cdots$
such that $n_1 < n_2 < n_3 \cdots$

$\therefore n_1$ is a peak point and $n_2 > n_1$

$\therefore a_{n_2} < a_{n_1}$

$\therefore n_2$ is not a peak point & $n_3 > n_2$

$\therefore a_{n_3} < a_{n_2}$

Repeating the above process we get a sub-sequence $\{a_{n_k}\}$ so that $a_{n_1} > a_{n_2} > a_{n_3} \cdots$

Thus the sequence $\{a_n\}$ contains a monotonically decreasing sequence $\{a_{n_k}\}$

Theorem (Bolzano Weierstrass)

Every bounded real sequence has a convergent subsequence

Proof # Let $\{a_n\}$ be bounded sequence.

Then there is a closed interval $I_0 = [a, b]$ such that

$$a_n \in [a, b] = I_0$$
$$\forall A \quad (\forall n) = I_0$$

Bisecting $I_0 = [a, b]$ into two equal intervals

$$\left[a, \frac{a+b}{2}\right], \left[\frac{a+b}{2}, b\right]$$

one of these intervals must contain a_n for infinitely many +ve integer. Let this interval be I_1

$$I_1 = \left[a, \frac{a+b}{2}\right] = \left[\frac{a+b}{2}, b\right] \text{ with length } \frac{a+b}{2} - a = \frac{b-a}{2}$$

and $I_0 \supset I_1$

Bisecting $I_1 = [a_1, b_1]$, 162 into two equal intervals we get

$$[a_1, \frac{a_1+b_1}{2}], \quad [\frac{a_1+b_1}{2}, b_1]$$

one of these contain infinite terms of the sequence $\{a_n\}$. Let this be

$$I_2 = [a_1, \frac{a_1+b_1}{2}] \quad \text{with length} = \frac{b_1 - a_1}{2} = \frac{b-a}{2^2}$$

and $I_1 \supset I_2$

Continuing this process, we obtain a sequence of nested intervals $I_0, I_1, I_2 - - -$ such that

$$I_0 \supset I_1 \supset I_2 \supset I_3 \supset - - - -$$

with $b_k - a_k = \frac{b-a}{2^k} \longrightarrow 0$ as $n \to \infty$

and each interval contain an infinite no of terms of the sequence.

Choose a +ve integer n_1 such that
$$a_{n_1} \in I_1$$

$\therefore$ $I_2 = [a_2, b_2]$ contain infinite terms of $\{a_n\}$

$\therefore$ $\exists$ +ve integer $n_2 > n_1$, such that
$$a_{n_2} \in I_2$$

Continuing this process we obtain numbers
$$a_{n_1}, a_{n_2}, a_{n_3} - - - - - - \text{ such that}$$
$$a_{n_k} \in [a_k, b_k] \quad k = 1, 2, 3 - - -$$

and $n_1 < n_2 < n_3 < n_4 - - - - -$

$\Rightarrow$ peak points

and the term (a_m) is called
a_m is never exceeded by any
i.e. a_m is in the sequence it in the sequence
sequence it follows

Proof :- $\{a_n\}$ is bounded

$\therefore$ By B.W Theorem there is

a convergent sub-sequence $\{a_{n_k}\}$.

Also $a_{n_k} \to l$ $\forall$ $n_k > n_0$

Hence $\lim_{n\to\infty} a_{n_k} \to l$

Cantor's Intersection Theorem.

For real line.

Theorem # of $F = \{F_n\}$ is a countable

class/family of non-empty closed and bounded

sets such that

$$F_1 \supset F_2 \supset F_3 \supset F_4 \cdots \supset F_n \supset \cdots$$

Then $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

Proof # :- each F_n is a non-empty

closed and bounded set

$\therefore$ $\exists$ exists sequences of real

nos M_n & m_n belong to F_n

such that

$M_n = \text{Sup } F_n$ $m_n = \text{Inf } F_n$ in A

$\therefore$ $F_{n+1} \subset F_n$ in A

$\therefore$ $M_n \ge M_{n+1}$ & $m_n \le m_{n+1}$ $\forall n \in \mathbb{N}$

Thus $\{a_{n_k}\}$ is a subsequence of a sequence $\{a_n\}$

we show that $\{a_{n_k}\}$ is cgt.

The sequence $\{a_n\}$ is monotone and bounded and so is cgt.

Let $\ell = \lim_{k\to\infty} c_k$.

Similarly the sequence $\{d_k\}$ converges some no m

$$m - \ell = \lim_{k\to\infty} d_k - \lim_{k\to\infty} c_k$$
$$= \lim_{k\to\infty}(d_k - c_k) = \lim_{k\to\infty} \frac{d-c}{2^k} = 0$$
$$\therefore \quad m = \ell$$

$a_{n_k} \in [c_k, d_k] \qquad k = 1, 2, 3, \dots$

$c_k \le a_{n_k} \le d_k$.

By squeeze Theorem

$$\lim_{k\to\infty} a_{n_k} = \ell = m$$

Thus $\{a_n\}$ contains a convergent subsequence

<u>Corollary</u> Suppose that $\{a_n\}$ is bounded sequence such that $a_n > \alpha$ if $n > n_0$ where α is a real number, then there is a convergent subsequence $\{a_{n_k}\}$ such that

$$\alpha \le \lim_{k\to\infty} a_{n_k}$$

...ence ...

that follows it in the sequence

Now the lowest bound f_n $\bigcap\limits_{n=1}^{\infty} F_n$ is the lower bound of the sequence $\{M_n\}$ of upper bounds. Thus $\{M_n\}$ is non-increasing sequence which is bounded below and is therefore convergent

Let $\lim\limits_{n\to\infty} M_n \to M$.

we show that $M \in \bigcap\limits_{n=1}^{\infty} F_n$

Let $M \notin \bigcap\limits_{n=1}^{\infty} F_n$

Then there will be at least one neighbourhood say $]M-\epsilon, M+\epsilon[$ $\epsilon > 0$ which contains no point of $\bigcap\limits_{n=1}^{\infty} F_n$

$\Rightarrow]M-\epsilon, M+\epsilon[$ contains no point of F_n F_n

(Some value of n say m

$\Rightarrow]M-\epsilon, M+\epsilon[$ contains no point of F_n, F_n $n > m$

$\Rightarrow M_n \notin]M-\epsilon, M+\epsilon[$ $\forall n > m$

Contradicting the fact that $\{M_n\}$ converges to M

Hence $M \in \bigcap\limits_{n\geq1}^{\infty} F_n$

Cluster points (Limit points) of a sequence

Frequently valid property # A produced property $P(n)$ is frequently valid. of statement $P(n)$ is frequently valid.

For a sequence $\{a_n\}$ if for every natural no m $\exists$ at least one m, $> m$ such that $P(n)$ is true.

<u>Note</u> An eventually valid statement is also frequently valid.

Cluster Point#

A real no c is said to be a cluster point of a sequence $\{a_n\}$ if every nbd of c contains infinitely many terms of the sequence. i.e.

$\forall \varepsilon > 0$ $\quad a_n \in (c-\varepsilon, c+\varepsilon)$ for infinitely many values of n.

<u>Note</u> A cluster point of a sequence is called a limit point or a condensation point or an accumulation point or a subsequential limit of the sequence.

Difference b/w limit and limit point of a sequence

If $l \in R$ is the limit of a sequence $\{a_n\}$ then for $\varepsilon > 0$ $\exists$ $m \in N$ such that

$$|a_n - l| < \varepsilon \qquad \forall n > m$$

$\Rightarrow$ every nbd of l contains all except a finite no of terms of the sequence. i.e. there are

$$\lim_{n \to \infty} a_n = l$$

168

only a finite no. of terms outside each nbd of l.
whereas if l is a limit point of sequence $\{a_n\}$, then every nbd of l contains infinitely many terms of the sequence and there may be infinite terms out the nbd i.e it does not exclude the possibility of an infinite no. of terms of sequence lying outside the interval or nbd

Hence limit of a sequence is a limit of the sequence but a limit point of a sequence need not be the limit of the sequence. e.g
$\{(-1)^n\}$ has limit points or cluster point
$1, -1$ but has no limit

<u>Note #</u> (1) If $a_n = l$ for infinitely many values of n, then l is a limit point of $\{a_n\}$

(2) If for an $\varepsilon > 0$, $(l-\varepsilon, l+\varepsilon)$, for finitely many values of n, then l can not be a cluster point of $\{a_n\}$

(3) Limit point of a sequence need not be a sequence.

<u>Def</u> : A real no l is called a cluster point of a sequence $\{a_n\}$ if given $\varepsilon > 0$ and a +ve integer m $\exists$ a no $k > m$ s.t
$$|a_k - l| < \varepsilon$$

Thus every nbd of l contains a term of the sequence. This is equivalent to saying that

every nbd of l contains infinitely many terms of the sequence because if $(l-\varepsilon, l+\varepsilon)$ contains only finite no of terms of the seq say

$a_1, a_2, ---- a_n$

Let $\delta = \min\{|l-a_1|, |l-a_2| ... |l-a_n|\}$

now $(l-\delta, l+\delta)$ contains no term of the sequence which is a contradiction.

Clearly above two definitions are equivalent

Def 3 A real no l is called a cluster point of a sequence $\{a_n\}$ if l is limit of some subsequence of $\{a_n\}$.

Note Cluster point is also called a subsequential limit

Limit point & range set & cluster point

We note that limit point of the range set $S = \{a_1, a_2, a_3 ---\}$ is automatically a cluster point of $\{a_n\}$. The two notions differ only in that for limit points the nbd is deleted. The distinction that for cluster point it is not. The two notions were introduced to cover the possibility that terms of a sequence may be repeated frequently and a sequence may be finite and has no limit point whereas sequence has infinite terms the range set may be finite and may have cluster point e.g for sequence $\{-1, 1\}$ has no limit point and may have the range set $\{-1, 1\}$ has no limit point $-1, 1$ are cluster point. Thus if a sequence $\{(-1)^n\}$

has a limit point L then L is a cluster point but the converse may not be usually true e.g.
$\{(-1)^n\}$ has $+1, -1$ cluster points but has no limit point.

Example 1 → 0 is a limit point of the sequence

Sol: $\{\frac{1}{n}\}$

For $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ s.t. that $\frac{1}{m} < \varepsilon$

∴ for $n > m$, $0 < \frac{1}{n} \leq \frac{1}{m} < \varepsilon$

⟹ $-\varepsilon < 0 < \frac{1}{n} < \varepsilon$ $\forall n > m$

⟹ $\frac{1}{n} \in (-\varepsilon, \varepsilon)$ $\forall n > m$

⟹ Every nbd of 0 contains infinitely many terms of the sequence $\{\frac{1}{n}\}$

⟹ 0 is a limit point

Example 2# The sequence $\{(-1)^n\}$ has two limit points

Example #3# The sequence $\{n\}$ has no cluster point

Theorem # If l is a limit point of the range of a sequence $\{a_n\}$, then l is a limit point of the sequence $\{a_n\}$

Proof # Lets = range $\{a_n\} = \{a_n : n \in \mathbb{N}\}$

∵ l is a limit point of s

∴ Every deleted nbd of s which are terms of $\{a_n\}$ contains infinitely many elements of s

⟹ Every nbd of s contains infinitely many terms of $\{a_n\}$

⟹ l is a limit point of the sequence $\{a_n\}$

Note # Converse of the above theorem may not be true

Consider

$$a_n = 1 + (-1)^n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even} \end{cases}$$

0, 2 are the limit points of the sequence.
But the range = {0, 2} is a finite set and Finite set has no limit point.

(2) If the terms of the sequence are distinct, then the limit points of the sequence are the limit points of the range set.

Theorem # If the limit points of a sequence converges to l, then l is the only limit point of the sequence.

Proof # The sequence {a_n} converges to l

⟹ Given $\epsilon > 0$ ∃ +ve integer m such that

$$|a_n - l| < \epsilon \qquad \forall n \geq m \to ①$$

⟹ $l - \epsilon < a_n < l + \epsilon \qquad \forall n \geq m$

⟹ $a_n \in (l-\epsilon, l+\epsilon)$ for infinitely many values of n.

⟹ every nbd of l contains infinitely many terms of the sequence {a_n}

⟹ l is a limit point of the sequence {a_n}

If possible let l' be another limit point of the sequence {a_n}
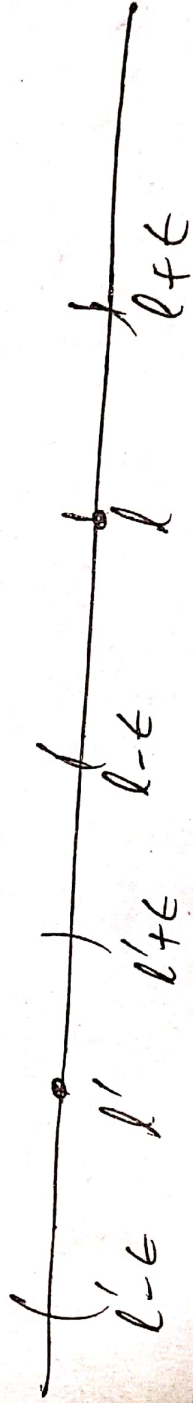
$l-\epsilon \qquad l \qquad l+\epsilon \qquad l'-\epsilon \qquad l' \qquad l'+\epsilon$

... term that follows ...

Let $\epsilon = \frac{1}{3}(l - l')$ where $l > l'$

Then $(l'-\epsilon, l'+\epsilon) \cap (l-\epsilon, l+\epsilon) = \phi$

From ①, $a_n \in (l-\epsilon, l+\epsilon)$ $\forall n \geq m$

$\therefore a_n \in (l'-\epsilon, l'+\epsilon)$ for almost all values of n

$\Rightarrow$ finitely many terms say a_n lie in $(l'-\epsilon, l'+\epsilon)$

Hence l' is not a limit point of the sequence

Hence l is only limit point of the sequence.

Theorem (Bolzano Weirshas theorem.)

Every bounded sequence has at least one limit point

Proof # Let $\{a_n\}$ be a bounded sequence.
and S is the range ic

$$S = \{a_n : n \in \mathbb{N}\}$$

$\therefore \{a_n\}$ is bounded.

$\therefore S$ is bounded.

Case I # Let S be a finite set

The $\exists$ a real no l such that $a_n = l$
for any infinite no of values $\exists n \in \mathbb{N}$.

$\Rightarrow$ Given $\epsilon > 0$, $a_n \in (l-\epsilon, l+\epsilon)$ for an infinite no of values $\exists n$

$\Rightarrow$ Every nbd of l contains infinitely many terms of the sequence $\{a_n\}$

$\Rightarrow l$ is a limit point of the sequence $\{a_n\}$

Case II Let S be an infinite set

Since S is an infinite bounded set, by B.W theorem for sets S has at least one limit point say l

Now l is a limit point of S

⟹ Every nbd of l contains an infinite no of elements of S

But each term of S is is a term of $\{a_n\}$

∴ Every nbd of l contains an infinite no of terms of the sequence $\{a_n\}$

⟹ l is a limit point of the sequence $\{a_n\}$

__Corollary #__ If S is a compact set and bounded.

(i.e compact) set, then every sequence in S has a limit point

__Proof #__ A sequence $\{a_n\}$ in S if $a_n \in S$ ∀n

Let $\{a_n\}$ be a sequence in S, then $a_n \in S$ ∀n

Since S is bounded, the sequence $\{a_n\}$ is bounded and consequently it has a limit point say l by B.W theorem.

We show that $l \in S$

Let $l \in S^c$, then S being closed, S^c is open

∴ S^c is nbd of l

But S^c contains no term of $\{a_n\}$. This contradicts the fact that l is a limit point of $\{a_n\}$

∴ $l \notin S^c$. Hence $l \in S$

Corollary 2 # of [73] I is a closed interval, then every sequence in I has a limit point in I

Proof # ∵ I is closed interval

∴ I is closed and bounded.

⟹ The result follows from Corollary 1.

Theorem # The set of limit points of a bounded sequence is bounded.

Proof Let $\{a_n\}$ be a bounded sequence

⟹ ∃ real no $k \not< K$ ($k \leq K$)

Such that

$$k \leq a_n \leq K \quad \forall n \in N$$

∴ $a_n \notin (-\infty, k) \not\subset a_n \notin (K, \infty)$

for any n

Let l be any real no

If $l \in (-\infty, k)$, then $(-\infty, k)$ contains no terms of the sequence & l is not limit point

If $l \in (K, \infty)$, Then (K, ∞) contains no term of sequence $\{a_n\}$ and l is not limit point of $\{a_n\}$

Thus no point outside $[l, K]$ is a limit point of $\{a_n\}$

⟹ the limit point of $\{a_n\}$ lie in $[l, K]$

⟹ The set of all the limit points of a bounded sequence is bounded.

Note The bounds of the set of limit points of a bounded sequence are same as the bounds of a bounded sequence

(2) The set of limit points of an unbounded sequence may or may not be bounded.

e.g $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ is unbounded.

but the set of limit points $\{0\}$ is bounded.

The sequence $\{2, 1+\frac{1}{2}, 2+\frac{1}{2}, 1+\frac{1}{3}, 2+\frac{1}{3}, \cdots\}$ is unbounded and the set of limit points is N which is unbounded.

__Theorem #__ Every bounded sequence has its greatest and the least limit points.

__Proof #__ Let $\{a_n\}$ be a bounded sequence.

Then $S = \{a_n\}$ the set E of limit points is also bounded. and $E \neq \phi$ (B.W. Theorem)

By Completeness property, E has minimum and supremum.

Let $\inf E = l$ & $Sup\ E = u$.

For $\varepsilon > 0$ let $(u-\varepsilon, u+\varepsilon)$ be nbd of u.

$Sup\ E = u \Rightarrow \exists$ some $x \in E$ s. That

$u-\varepsilon < x \leq u < u+\varepsilon$

$\Rightarrow x \in (u-\varepsilon, u+\varepsilon)$

$\Rightarrow (u-\varepsilon, u+\varepsilon)$ is a nbd of x

$\therefore x \in E$ is a limit point of $\{a_n\}$

$\Rightarrow$ every nbd of x contains infinitely many terms of $\{a_n\}$

$\Rightarrow (u-\varepsilon, u+\varepsilon)$ contains infinitely many terms of $\{a_n\}$

This is true for every $\varepsilon > 0$

$\Rightarrow u$ is a limit point of $\{a_n\}$

$\Rightarrow u \in E$

Similarly $l \in E$.

Theorem # The set of limit points of a bounded sequence is compact set.

Proof # Let E be the set of all limit points of a bounded sequence $\{a_n\}$.

Then E is closed and bounded. Subset of R.

$\Rightarrow$ E is compact

The Generalised limits (Upper and lower limits)

We have discussed limit of a Cgt sequence and have also proved that bounded monotone sequence always converge. There are sequences which are bounded but not monotone. Such sequences can converge but can equally well diverge as

$$\{(-1)^{n+1}\}$$

From this example we note that a general bounded sequence can diverge by oscillating between various limits. This oscillation suggests trigonometric function.

The Lim sup & Lim Inf are defined for arbitrary (not necessarily cgt) sequences

If $\{a_n\}$ is bounded, then by B.W theorem $\{a_n\}$ has a cgt subsequence. The no $\lim \sup a_n$ $n \to \infty$ is the max value obtainable as the limit of cgt subsequence of $\{a_n\}$ i.e maximum of all limit points (cluster points of a sequence and $\lim \text{Inf} a_n$ $n \to \infty$ is the minimum value obtainable as the limit of a cgt subsequence of $\{a_n\}$ i.e min of all

$\lim_{n\to\infty} \sup a_n$ & $\lim_{n\to\infty} \inf a_n$ are also

denoted by

$\overline{\lim_{n\to\infty}} a_n$ & $\underline{\lim_{n\to\infty}} a_n$.

We discuss these limits under two categories.

(a) For bounded sequences (b) for unbounded sequences.

Lim Sup & Lim Inf of Bounded Sequences

Let $\{a_n\}$ be a bounded sequence and E be the set of all limit points of $\{a_n\}$. Then,

$$\lim_{n\to\infty} \sup a_n = \text{lub } E = \text{Sup } E$$
$$\lim_{n\to\infty} \inf a_n = \text{glb } E = \text{Inf } E$$

Thus $\lim_{n\to\infty} \sup a_n = \mu$

if for every ϵ

$$|a_n - \mu| < \epsilon$$

for infinitely many values of n and no no larger than μ has this property.

$\lim_{n\to\infty} \inf a_n = \nu$

if for every $\epsilon > 0$

$$|a_n - \nu| < \epsilon$$

for infinitely many values of n & no number less than ν has this property.

is true for every $\epsilon > 0$

ν is a limit point of $\{a_n\}$

$\nu \in E$.

$\ell \in E$.

Theorem # Let $\{a_n\}$ be a bounded sequence. Then

$$\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$$

Proof # Let E be the set of Subsequential limits of $\{a_n\}$

By definition

$$\text{Inf } E \leq \text{Sup } E$$

$$\Rightarrow \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$$

Theorem # A real no U is the limit superior of a bounded sequence $\{a_n\}$ iff

(a) For each $\epsilon > 0$, $a_n > U - \epsilon$ for infinitely many values of n.

(b) For each $\epsilon > 0$, $a_n < U + \epsilon$ for all except for finitely many values of n.

Proof # **Necessity**

Let U be $\lim$ Superior of $\{a_n\}$ & let $\epsilon > 0$ be given.

$\therefore U$ is a limit point / cluster point of a_n.

$\therefore a_n \in (U - \epsilon, U + \epsilon)$ For infinitely many values of n

In particular $a_n > U - \epsilon$ For infinitely many values

Again Since U is the greatest limit point, $U + \epsilon$ is not a limit point and therefore many values of n

$a_n \geq U + \epsilon$ for infinite values of n, then

(If for some $\epsilon > 0$, $a_n \geq U + \epsilon$ for infinite values of n, then $\{a_n\}$ will have limit point $\geq U + \epsilon$

$\therefore a_n < U + \epsilon$ for all except finite values of n

Remarks #(1) For a cgt sequence all subsequences converge to same limit but a cgt bounded sequence has many cgt subsequences. Lim sup & lim Inf give the behaviour of subsequences. Lim sup & lim Inf behaviour signifies how much the set of limit points. Thus rise or fall when n is very large enough. The set of limit points for a cgt sequence $\{a_n\}$ can also bounded and hence lub, glb

(2) If $\{a_n\}$ is unbounded, $\lim\limits_{n\to\infty} \sup a_n = \infty$

and $\lim\limits_{n\to\infty} \inf a_n = -\infty$

Example (i) For sequence $\{(-1)^n\}$, the only limit points are $-1, 1$. So $E = \{-1, 1\}$

$\lim\limits_{n\to\infty} \inf a_n = -1$

$\lim\limits_{n\to\infty} \sup a_n = \sup E = 1$

(ii) For sequence $\{\frac{1}{n}\}$, the only limit point is $\{0\}$. So $\lim\limits_{n\to\infty} \sup a_n = \lim\limits_{n\to\infty} \inf a_n = 0$

(iii) If $a_n \geq k$ $\forall n$

$\lim \inf a_n = \lim \sup a_n \geq k$

(iv) If $a_n = \begin{cases} 2 & \text{when } n \text{ is odd} \\ -n & \text{when } n \text{ is even} \end{cases}$

Then 2 is limit point of $\{a_n\}$ which is unbounded below. $\underline{\lim} a_n = -\infty$ $\overline{\lim} a_n \geq 2$

(v) For sequence $a_n = (-1)^n n$ $\forall n \in N$.

$\lim\limits_{n\to\infty} \inf a_n = -\infty$ $\lim\limits_{n\to\infty} \sup a_n = \infty$

Sufficiency

Given $\epsilon > 0$, $u-\epsilon < a_n$ for infinitely many values of n and $u+\epsilon > a_n$ for all except finitely many values of n

$\Rightarrow u-\epsilon < a_n < u+\epsilon$ for infinitely many values of n

$\Rightarrow u$ is a limit point of $\{a_n\}$

Now we show that no no greater than u can be limit point of $\{a_n\}$.

Let u' be any other limit point of $\{a_n\}$.

Let p, q be two numbers such that

$$u < p < u' < q$$

By 2nd condition, for each $\epsilon > 0$, $a_n < u+\epsilon$ for all except for finite values of n

Choosing $p-u = \epsilon > 0$, we have.

$$a_n < p \quad \text{For all except finitely many values of } n$$

and therefore (p, q) is a nbd of u' containing a_n for finitely many values of $n \Rightarrow u'$ is not a limit point of $\{a_n\}$ and u is the greatest limit point of $\{a_n\}$

Hence u is limit superior of $\{a_n\}$

Theorem

A real no l is the limit inferior of a bounded sequence $\{a_n\}$ iff the following are true.

(i) for each $\epsilon > 0$, $a_n < l+\epsilon$ for infinitely many values of n

(ii) for each $\epsilon > 0$, $a_n > l-\epsilon$ for all except finitely many values of n

Proof

Necessity

Let l be limit inferior of $\{a_n\}$ and $\epsilon > 0$ be given

∴ l is limit inferior of $\{a_n\}$

∴ $l - \epsilon < a_n < l + \epsilon$ for infinitely many n

In particular $a_n < l + \epsilon$ for infinitely many values of n

Again since l is the least limit point, $l - \epsilon$ is not a limit point and.

$a_n \le l - \epsilon$ for finitely many values of n

because if $\epsilon > 0$, $a_n \le l - \epsilon$ for infinitely many values of n, then, then $\{a_n\}$ will have a limit point $P \le l - \epsilon$

∴ $a_n > l - \epsilon$ for all except finitely many values of n

Sufficiency

Let us assume that l satisfies both the conditions

Given $\epsilon > 0$, $a_n < l + \epsilon$ for infinite values of n

and

$a_n > l - \epsilon$ for all except finite values of n

⟹ $l - \epsilon < a_n < l + \epsilon$ for infinitely many values of n

∴ l is a limit point of $\{a_n\}$

We show that no number less than l can be a limit point of $\{a_n\}$

Let l' be any number less than l

(18)

Let P, Q be two members such that

$$P < l' < l < l$$

By 2nd condition. For each $\varepsilon > 0$

$$a_n > l - \varepsilon \text{ for all except for finite values of } n$$

$\Rightarrow$ For $\varepsilon = l - l' > 0$, we have,

$$a_n > l - \varepsilon = l - (l - l') = l' \text{ for all except finite values of } n$$

$\Rightarrow (l', \delta)$ is a nbd of (say) l' containing a_n
for finitely many values of n

$\Rightarrow l'$ is not a limit point of $\{a_n\}$ so that
l is the least limit point of $\{a_n\}$ and hence.
l is limit inferior of $\{a_n\}$

Theorem

A sequence $\{a_n\}$ converges to l iff

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \inf a_n = l$$

Proof

Let the sequence $\{a_n\}$ converges to l
Then given $\varepsilon > 0$ $\exists$ +ve integer m such that

$$\qquad\qquad \forall n \geq m$$

$$|a_n - l| < \varepsilon \quad \forall n \geq m$$

$$\Rightarrow l - \varepsilon < a_n < l + \varepsilon \quad \forall n \geq m$$

Since the nbd $(l-\varepsilon, l+\varepsilon)$ of l contains a_n for
infinitely many values of n and since ε is arbitrary
therefore every nbd of l contains infinitely many terms of
the sequence $\{a_n\}$

$\therefore l$ is a limit point of $\{a_n\}$

We show that l is only limit point of $\{a_n\}$

$\Rightarrow \lim_{n \to \infty} a_n = l \Rightarrow \{a_n\}$ has any one cluster point

Let l' be any number $l \& l^2$ other than l. Two cases arise

(i) $l < l'$ (ii) $l' < l$

Suppose $l < l'$. Let P, V, λ be three numbers such that $P < l < V < l' < \lambda$

∴ $a_n \to l$

∴ Every nbd of l contains a_n for all except finitely many values of n. In particular (P.V) for finitely many values of n all except finitely many values a_n for almost all except finitely many values a_n for almost

⇒ The nbd (S, λ) of l' contains a_n of finite values of n

⇒ l' cannot be a limit point of $\{a_n\}$

⇒ l' is not a limit point of $\{a_n\}$

Similarly $l' < l$, l' is not a limit point of $\{a_n\}$

∴ Ans. Thus l is any limit point of $\{a_n\}$

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n$$

Hence $\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n$

__Converse__ Let $\varepsilon > 0$ be given.

∴ $l = \lim_{n\to\infty} \sup a_n$

∴ $a_n < l + \varepsilon$ for all except finitely many

⇒ ∃ a +ve integer m_1 s.t

$a_n < l + \varepsilon$ $\forall n \geq m_1$

Again ∴ $l = \lim_{n\to\infty} \inf a_n$

$a_n > l - \varepsilon$ for all except finitely many values of n

∴ ∃ a +ve integer m_2 such that

⇒ $a_n > l - \varepsilon$ $\forall n \geq m_2$

Limit Superior and Limit Inferior (For bounded sequence)

Let $\{a_n\}$ be bounded sequence and bounded above by K. Then for each $n \in \mathbb{N}$ the

set $S_n = \{a_n, a_{n+1}, \dots\}$ is bounded above by K. By $l.u.b$ axiom S_n has $l.u.b$ M_n.

i.e.

$$M_n = \sup S_n = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

clearly $\qquad M_n \geqslant M_{n+1} \qquad \forall n \in \mathbb{N}$

Thus the sequence $\{M_n\}$ being decreasing sequence either converges or diverges to $-\infty$.

If $\{M_n\}$ is convergent, then $\lim\limits_{n \to \infty} M_n$ is called limit superior of $\{a_n\}$ i.e.

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} M_n$$

If $\{a_n\}$ not bounded above we can set $M_n \geqslant 0$

$$\lim_{n \to \infty} \sup a_n = \lim_{n \to \infty} \sup a_n = +\infty$$

If $\{M_n\}$ is divergent then $\lim\limits_{n \to \infty}$

Let $\{a_n\}$ be bounded below by K. Then for each n

each n

$$S_n = \{a_n, a_{n+1}, \dots\} \text{ is bounded below}$$

by K for each n

$\Rightarrow$ S_n has g.l.b m. Let

clearly $m \leq x_{n+1}$ $\forall n \in N$.

Thus sequence $\{m_n\}$ being an increasing

sequence is either convergent or diverges to $+\infty$

$\therefore$ $\{m_n\}$ is cgt, then

$$\lim_{n\to\infty} \inf a_n = \lim m_n = \lim \inf S_n$$

If $\{a_n\}$ is not bdded below we define $\lim \inf \{a_n, a_{n+1}, \ldots\}$

a sequence $\{a_n\}$ is such that

$$\lim_{n\to\infty} a_n = \infty$$

, then $\{a_n\}$ diverges to ∞

Theorem #

(a) If sequence $\{a_n\}$ is such that

$$\lim_{n\to\infty} a_n = \infty \text{, then } \{a_n\}$$

(b) If a sequence $\{a_n\}$ is such that

$$\lim_{n\to\infty} a_n = -\infty \text{, then } \{a_n\}$$

diverges.

Proof # (a) Let $M_n = \sup \{a_n, a_{n+1}, a_{n+2} \ldots\}$

and $m_n = \inf \{a_n, a_{n+1}, a_{n+2} \ldots\}$

Then $m_n \leq a_n \leq M_n$ $\forall n$.

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} a_n = \infty \quad \& \quad \lim_{n\to\infty} M_n = \lim_{n\to\infty} a_n$$

Since

$$= \infty \longrightarrow ①$$

From ① $\lim_{n\to\infty} a_n = \infty$

The sequence diverges to ∞

(b) $\therefore$ $\lim_{n\to\infty} m_n = \lim_{n\to\infty} a_n = -\infty = \lim_{n\to\infty} a_n$

$\Leftarrow$ $\lim_{n\to\infty} a_n = -\infty$

$\Rightarrow$ $\{a_n\}$ diverges to $-\infty$

$\Rightarrow$ $a_n \geq l-\varepsilon$

Examples

Q.1 Give examples of sequences having.
(i) no cluster point   (ii) one cluster point
(iii) Two cluster points   (iv) infinitely many cluster points.

Sol:
(i) sequence $\{n\}$ has no cluster point
(ii) The sequence $\{n\}$ has one cluster point
(iii) The sequence $\{(-1)^n\}$ has two cluster points namely $-1, 1$
(iv) The sequence $\{2, 1+\frac{1}{2}, 2+\frac{1}{2}, 1+\frac{1}{3}, 2+\frac{1}{3}, 3+\frac{1}{3}, 1+\frac{1}{4}, 2+\frac{1}{4}, 3+\frac{1}{4}, \cdots\}$

has infinitely many cluster points. Every natural no is a cluster point

Q.2 Find cluster points of the sequences defined by nth term

(i) $(-1)^n$ (ii) 5 (iii) $\dfrac{(-1)^n}{n}$

(iv) n (v) $(-1)^n\left(1+\frac{1}{n}\right)$ (vi) $\left(1+\frac{1}{n}\right)^{n+1}$

(vii) $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

Sol:
(i) $a_n = (-1)^n = \begin{cases} -1 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even,} \end{cases}$

[Ans two cluster points.]

(ii) $a_n = 5$ is Constant sequence Converging to 5
→ $\{a_n\}$ has only one cluster point

(iii) $a_n = \dfrac{(-1)^n}{n} = \begin{cases} -1/n & \text{when } n \text{ is odd.} \\ 1/n & \text{when } n \text{ is even.} \end{cases}$

for $a_n \xrightarrow[n\to\infty]{} 0 \Rightarrow \{a_n\}$ has only one cluster point

(iv) for any $\ell \in R$ $\left(\ell - \frac{1}{n}, \ell + \frac{1}{n}\right)$ Contains at most one term of the sequence $\{n\}$

$\therefore \ell \in R$ is not a limit point of $\{n\}$

$\Rightarrow$ The sequence $\{n\}$ has no limit point

(v)
$$a_n = (-1)^n\left(1+\frac{1}{n}\right) = \begin{cases} -\left(1+\frac{1}{n}\right) & n \text{ is odd} \\ \left(1+\frac{1}{n}\right) & n \text{ is even} \end{cases}$$

$\Rightarrow$ Sequence $\{a_n\}$ has two cluster points $-1, 1$

(vi) Here
$$a_n = \left(1+\frac{1}{n}\right)^{n+1}$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) = e \times 1 = e$$

$\Rightarrow$ sequence $\{a_n\}$ converges to e and has one cluster point

(vii)
$$a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

The sequence converges to e

$\therefore$ The sequence has only one cluster point

Q# Find the Limit Superior and limit inferior of each of the following sequences:

(i) $\{1,3,5,1,3,5,1,3,5,\cdots\}$

(ii) $\{1,5,17,17,1,5,17,19,1,5,17,19,13,\cdots\}$

(iii) $\{a_n\}$ where $a_n = \dfrac{5 \sin\frac{n\pi}{3}}{3}$

(iv) $\{a_n\}$ where $a_n = (-2)^{-n}\left(1+\frac{1}{n}\right)$

(v) $\{a_n\}$ where $a_n = (-10)^n\left(1+\frac{1}{n}\right)^2$

(vi) $\{a_n\}$ where $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

$\lim_{n\to\infty}$

$\Rightarrow \{a_n\}$ diverges to $-\infty$

$\Rightarrow$ I ... $a_n \geq \ell - \epsilon$

To BILASPUR (HINDI)

(87)

(vii) $\{a_n\}$ $a_n = \left(1 + \frac{1}{n}\right)^{n+1}$

(viii) $\{a_n\}$ $a_n = (-1)^n (2^n + 3^n)$

Sol (i) $a_n = \begin{cases} 1 & \text{if } n = 3m-2 \\ 3 & \text{if } n = 3m-1 \\ 5 & \text{if } n = 3m \end{cases}$ $m \in N$

The set of cluster points of $\{a_n\} = E = \{1, 3, 5\}$

$\overline{\lim_{n \to \infty}} a_n = Max \{1, 3, 5\} = 5$

$\underline{\lim_{n \to \infty}} a_n = min \{1, 3, 5\} = 1$

(ii) The set of cluster points = $E = \{1, 5, 17, 15\}$

$\overline{\lim_{n \to \infty}} a_n = Max\, E = 19$

$\underline{\lim_{n \to \infty}} a_n = Min\, E = 1$

(iii) $a_n = Sin \frac{n\pi}{3} = \begin{cases} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{cases}$

    if $n = 3m$,

    if $n = 6m$ or $6m-4$

    if $n = 6m-2$ or $6m-1$

      where $m \in N$

The set of cluster points of $\{a_n\} = E = \left\{ 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \right\}$

$\Rightarrow \overline{\lim_{n \to \infty}} a_n = \frac{\sqrt{3}}{2}$

$\underline{\lim_{n \to \infty}} a_n = -\frac{\sqrt{3}}{2}$

(iv) Have $\{a_n\}$ converges to 0 $\therefore E = \{0\}$

The set of cluster points = $\lim_{n \to \infty} a_n = 0$

$\Rightarrow \overline{\lim_{n \to \infty}} a_n = 0$

$$\Rightarrow \lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} a_n. \qquad \underline{188}$$

(v) Here $a_n = (-1)^n\left(1+\frac{1}{n}\right)^2 = \begin{cases} -(10)^n\left(1+\frac{1}{2}\right)^2 & n \text{ odd} \\ (10)^n\left(1+\frac{1}{n}\right)^2 & n \text{ even.} \end{cases}$

$$\therefore \lim_{n\to\infty}(10)^n\left(1+\frac{1}{2n}\right)^2 = \infty$$

$$E = \{-\infty, +\infty\}$$

$$\lim_{n\to\infty} a_n = \infty$$

$$\lim_{n\to\infty} a_n = -\infty$$

(vi) $a_n = (-1)^n\left(1-\frac{1}{n}\right) = \begin{cases} 1-\frac{1}{n} & n \text{ even.} \\ -1+\frac{1}{n} & n \text{ odd.} \end{cases}$

$$E = \{1, -1\}$$

$$\lim_{n\to\infty} a_n = -1$$

$$\lim_{n\to\infty} a_n = 1$$

(vii) $a_n = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)$

$$\lim_{n\to\infty} a_n = e \times 1 = e$$

$$\lim_{n\to\infty} a_n = e$$

$$\lim_{n\to\infty} a_n = e$$

(viii) $a_n = (-1)^n\left(\frac{2^n+3^n}{2^n+3^n}\right) = \begin{cases} -\left(\frac{2^n+3^n}{...}\right) & n \text{ odd} \\ \left(\frac{2^n+3^n}{...}\right) & n \text{ even.} \end{cases}$

$$E = \{-\infty, +\infty\}$$

$$\text{if } \lim_{n\to\infty} a_n = \infty$$

$$\lim_{n\to\infty} a_n = -\infty$$

(vi) $\{a_n\}$ where $a_n = (-1)^n\left(1-\frac{1}{n}\right)$

$\Rightarrow \{a_n\}$ diverges to $-\infty$

$\Rightarrow \lim_{n\to\infty} a_n \to l-e$

Theorem # For a sequence $\{a_n\}$ prove that

$$\underline{\lim_{n\to\infty}} a_n \leq \overline{\lim_{n\to\infty}} a_n$$

Proof # If $\{a_n\}$ is unbounded, then either

$$\overline{\lim_{n\to\infty}} a_n = \infty \quad \text{or} \quad \underline{\lim_{n\to\infty}} a_n = -\infty$$

and hence there is nothing to prove.

Let a_n be a bounded sequence.

Let $m_n = glb\{a_n, a_{n+1}, a_{n+2} \cdots \}$

$M_n = lub\{a_n, a_{n+1}, a_{n+2} \cdots \}$ $\quad \forall n \in N.$

Then $m_n \leq M_n$.

$\Rightarrow \varliminf_{n\to\infty} m_n \leq \varliminf_{n\to\infty} M_n$

$\Rightarrow \underline{\lim_{n\to\infty}} a_n \leq \overline{\lim_{n\to\infty}} a_n$

Theorem # If $\{a_n\}$ & $\{b_n\}$ are two sequences such that $a_n \leq b_n \quad \forall n \in N,$ then

(i) $\underline{\lim_{n\to\infty}} a_n \leq \underline{\lim_{n\to\infty}} b_n$ 　(ii) $\overline{\lim_{n\to\infty}} a_n \leq \overline{\lim_{n\to\infty}} b_n$

Proof # Let $M_n = lub\{a_n, a_{n+1}, a_{n+2} \cdots \}$

$M_n' = lub\{b_n, b_{n+1}, \cdots \}$

$m_n = glb\{a_n, a_{n+1}, \cdots \}$

$m_n' = glb\{b_n, b_{n+1} \cdots \}$

$\underline{\lim_{n\to\infty}} a_n + \overline{\lim_{n\to\infty}} b_n$

$$\therefore \ a_n \leq b_n$$

$$\therefore \ M_n \geq M_n'$$

$$\therefore \ \overline{\lim_{n\to\infty}} M_n \geq \overline{\lim_{n\to\infty}} M_n', \quad \forall n$$

$$\Leftarrow \ \overline{\lim_{n\to\infty}} a_n \leq \overline{\lim_{n\to\infty}} b_n$$

$$\Leftarrow \ \underline{\lim_{n\to\infty}} a_n \leq \underline{\lim_{n\to\infty}} b_n$$

Theorem # If $\{a_n\}$ & $\{b_n\}$ are bounded sequences, then show that

(i) $\overline{\lim_{n\to\infty}} (a_n + b_n) \leq \overline{\lim_{n\to\infty}} a_n + \overline{\lim_{n\to\infty}} b_n$

(ii) $\underline{\lim_{n\to\infty}} (a_n + b_n) \geq \underline{\lim_{n\to\infty}} a_n + \underline{\lim_{n\to\infty}} b_n$

Proof # Let $M_n = \sup\{a_n, a_{n+1}, a_{n+2} \cdots\}$

$$M_n' = \sup\{b_n, b_{n+1}, \cdots\}$$

$$m_n = \inf\{a_n, a_{n+1} \cdots\}$$

$$m_n' = \inf\{b_n, b_{n+1}, \cdots\}$$

$\because \ \{a_n\}$ & $\{b_n\}$ are bounded.

$\therefore \ \{a_n + b_n\}$ is bounded.

$\therefore \ \sup\{a_n+b_n, \ a_{n+1}+b_{n+1} \cdots\} \leq \sup\{a_n, a_{n+1} \cdots\}$
$$+ \sup\{b_n, b_{n+1} \cdots\}$$
$$= M_n + M_n'$$

$\inf\{a_n+b_n, \ a_{n+1}+b_{n+1} \cdots\} \geq \inf\{a_n, a_{n+1} \cdots\} + \inf\{b_n, b_{n+1} \cdots\}$

(i) $\overline{\lim_{n\to\infty}}(a_n+b_n) = \lim_{n\to\infty} \sup\{a_n+b_n, \ a_{n+1}+b_{n+1} \cdots\}$

$$\leq \lim_{n\to\infty} (M_n + M_n')$$

$\{a_n\}$ diverges to $-\infty$

$a_n > l - \epsilon$

$$= \underline{\lim_{n\to\infty}} M_n + \underline{\lim_{n\to\infty}} M_n{}'$$

$$= \underline{\lim_{n\to\infty}} a_n + \underline{\lim_{n\to\infty}} b_n$$

(ii)
$$\overline{\lim_{n\to\infty}} (a_n + b_n) \geq \overline{\lim_{n\to\infty}} (M_n + M_n{}')$$

$$\geq \overline{\lim_{n\to\infty}} M_n + \overline{\lim_{n\to\infty}} M_n{}'$$

$$= \overline{\lim_{n\to\infty}} a_n + \overline{\lim_{n\to\infty}} b_n$$

Note(1) By combining the above two

$$\underline{\lim_{n\to\infty}} a_n + \underline{\lim_{n\to\infty}} b_n \leq \underline{\lim_{n\to\infty}} (a_n + b_n) \leq \overline{\lim_{n\to\infty}} (a_n + b_n)$$

$$\leq \overline{\lim_{n\to\infty}} a_n + \overline{\lim_{n\to\infty}} b_n$$

(2) In certain cases strict inequalities may hold.

e.g.

$$a_n = (-1)^n \qquad \& \quad b_n = (-1)^{n+1}$$

Then $\overline{\lim_{n\to\infty}} a_n = 1 \qquad \overline{\lim} b_n = 1$

$$\therefore \quad a_n + b_n = 0$$

$$\Rightarrow \overline{\lim_{n\to\infty}} (a_n + b_n) = 0 > \overline{\lim_{n\to\infty}} a_n + \underline{\lim_{n\to\infty}} b_n$$

$$\& \quad \underline{\lim_{n\to\infty}} (a_n + b_n) > \overline{\lim_{n\to\infty}} a_n + \underline{\lim_{n\to\infty}} b_n$$

Theorem # If $\{a_n\}$ is a bounded sequence, then

(i) $\varliminf_{n\to\infty} (-a_n) = -\varlimsup_{n\to\infty} a_n$

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(iii) $\varlimsup_{n\to\infty} (\lambda a_n) = \lambda \varlimsup_{n\to\infty} a_n$, $\quad \lambda > 0$

(iv) $\varliminf_{n\to\infty} (\lambda a_n) = \lambda \varliminf_{n\to\infty} a_n$, $\quad \lambda > 0$

(v) $\varlimsup_{n\to\infty} (\lambda a_n) = \lambda \varliminf_{n\to\infty} a_n$, $\quad \lambda > 0$

Proof # $\therefore$ $\{a_n\}$ is bounded.

$\therefore \{-a_n\}$ & $\{\lambda a_n\}$ are also bounded.

(i) $\varliminf_{n\to\infty} (-a_n) = \varlimsup_{r\to\infty} \sup \{-a_n, -a_{n+1} - - - \}$

$= \varliminf_{n\to\infty} - \inf \{a_n, a_{n+1}, - - - \}$

$= - \varlimsup_{n\to\infty} a_n$

(ii) $\varlimsup_{r\to\infty} (-a_n) = \varliminf_{r\to\infty} \inf \{-a_n, -a_{n+1}, - - - \}$

$= \varliminf_{n\to\infty} -\sup \{a_n, a_{n+1}, - - \}$

$= - \varlimsup_{n\to\infty} \sup \{a_n, a_{n+1} - - - \}$

$= -\varliminf_{n\to\infty} a_n$

(iii) $\varlimsup_{r\to\infty} (\lambda a_n) = \varliminf_{n\to\infty} \sup \{\lambda a_n, \lambda a_{n+1} - - - \}$

$= \lambda \varliminf_{n\to\infty} \sup \{a_n, a_{n+1} - - - \}$

$= \lambda \varlimsup_{n\to\infty} a_n$

Try others.

Subsequential Limit

A real no l is called a subsequential limit of a sequence $\{a_n\}$ if there is a subsequence of $\{a_n\}$ converging to $\{a_n\}$

A subsequential limit a sequence $\{a_n\}$ is also called a cluster point or limit point of the sequence.

Theorem# A real no l is called a subsequential limit of the sequence $\{a_n\}$ iff each neighbourhood $(l-\epsilon,\ l+\epsilon),\ \epsilon>0$ of l contains infinitely many terms of $\{a_n\}$ (i.e iff l is a cluster point of any $\{a_n\}$

Proof # Let l be a subsequential limit of $\{a_n\}$. Then $\exists$ a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ converging to l

Given $\epsilon>0$ $\exists$ even integer k_0 such that

$$|a_{n_k}-l|<\epsilon \qquad \forall\ k\geqslant k_0$$

$$\Rightarrow a_{n_k}\in(l-\epsilon,\ l+\epsilon) \qquad \forall\ k\geqslant k_0$$

$\Rightarrow$ Infinitely many terms of sequence $\{a_n\}$ & hence of $\{a_n\}$ lie in $(l+\epsilon,\ l-\epsilon)$

Converse Let each nbd of $(l-\epsilon,\ l+\epsilon)$ of l contains infinitely many terms of $\{a_n\}$

Then $a_n\in(l-\epsilon,\ l+\epsilon)$ for infinitely many n

In particular for $\epsilon_2 \frac{1}{n}$

$$a_n\in\left(l-\tfrac1n,\ l+\tfrac1n\right)\Rightarrow I_n$$

Choose $a_{n_1} \in (l-1, l+1)$. Then $\exists \; n_2 > n_1$, such that

$a_{n_2} \in (l-\frac{1}{2}, l+\frac{1}{2}) = I_2$

Continuing like this $\exists$ a natural no n_{k_0} s.t. that

$n_{k_0} > \cdots n_2 > n_1$ & $a_{n_{k_0}} \in (l-\frac{1}{k_0}, l+\frac{1}{k_0}) = I_{k_0}$

Again continuing in this way get a subsequence $\{a_{n_k}\}$ of $\{a_n\}$

Now for all $n_k \geq n_{k_0}$, we have $k \geq k_0$

$\Rightarrow \frac{1}{k} \leq \frac{1}{k_0}$

$\Rightarrow l + \frac{1}{k} \leq l + \frac{1}{k_0}$ & $l - \frac{1}{k} \geq l - \frac{1}{k_0}$

$\Rightarrow (l - \frac{1}{k}, l + \frac{1}{k}) \subset (l - \frac{1}{k_0}, l + \frac{1}{k_0})$

$\Rightarrow I_k \subset I_{k_0} \qquad \forall k \geq k_0$

$\Rightarrow \forall n_k \geq n_{k_0}, \; a_{n_k} \in I_k \Rightarrow a_{n_k} \in I_{k_0}$

$\Rightarrow a_{n_k} \in (l - \frac{1}{k_0}, l + \frac{1}{k_0}) \quad \forall n_k \geq n_{k_0}$

$\Rightarrow |a_{n_k} - l| < \frac{1}{k_0} = \epsilon \quad \forall n_k \geq n_{k_0}$

$\Rightarrow \{a_{n_k}\}$ converges to l

$\Rightarrow l$ is a subsequential limit of the sequence.

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